Section 7.3 Double-Angle, Half-Angle, and Sum-Product Identities

Double-Angle Formulas

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Formula for sine: $\sin 2x = 2\sin x\cos x$

Formulas for cosine: $\cos 2x = \cos^2 x - \sin^2 x$

$= 1 - 2\sin^2 x$

$= 2\cos^2 x - 1$

Formula for tangent: $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

EXAMPLE: If $\cos x = -\dfrac{2}{3}$ and x is in quadrant II, find $\cos 2x$ and $\sin 2x$.

Solution: Using one of the double-angle formulas for cosine, we get

$$\cos 2x = 2\cos^2 x - 1 = 2\left(-\frac{2}{3}\right)^2 - 1 = \frac{8}{9} - 1 = -\frac{1}{9}$$

To use the formula $\sin 2x = 2\sin x\cos x$, we need to find $\sin x$ first. We have

$$\sin x = \sqrt{1 - \cos^2 x} = \sqrt{1 - \left(-\frac{2}{3}\right)^2} = \frac{\sqrt{5}}{3}$$

where we have used the positive square root because $\sin x$ is positive in quadrant II. Thus

$$\sin 2x = 2\sin x\cos x = 2\left(\frac{\sqrt{5}}{3}\right)\left(-\frac{2}{3}\right) = -\frac{4\sqrt{5}}{9}$$

EXAMPLES:

(a) Write $\cos 3x$ in terms of $\cos x$.

(b) Prove the identity $\dfrac{\sin 3x}{\sin x\cos x} = 4\cos x - \sec x$.

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(a) Write $\cos 3x$ in terms of $\cos x$.

Solution: We have

$$\begin{aligned}
\cos 3x &= \cos(2x + x) \\
&= \cos 2x \cos x - \sin 2x \sin x \\
&= (2\cos^2 x - 1)\cos x - (2\sin x \cos x)\sin x \\
&= 2\cos^3 x - \cos x - 2\sin^2 x \cos x \\
&= 2\cos^3 x - \cos x - 2\cos x(1 - \cos^2 x) \\
&= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x \\
&= 4\cos^3 x - 3\cos x
\end{aligned}$$

(b) Prove the identity $\dfrac{\sin 3x}{\sin x \cos x} = 4\cos x - \sec x$.

Solution: We have

$$\begin{aligned}
\frac{\sin 3x}{\sin x \cos x} &= \frac{\sin(x + 2x)}{\sin x \cos x} \\
&= \frac{\sin x \cos 2x + \cos x \sin 2x}{\sin x \cos x} \\
&= \frac{\sin x(2\cos^2 x - 1) + \cos x(2\sin x \cos x)}{\sin x \cos x} \\
&= \frac{\sin x(2\cos^2 x - 1)}{\sin x \cos x} + \frac{\cos x(2\sin x \cos x)}{\sin x \cos x} \\
&= \frac{2\cos^2 x - 1}{\cos x} + 2\cos x \\
&= 2\cos x - \frac{1}{\cos x} + 2\cos x \\
&= 4\cos x - \sec x
\end{aligned}$$

Half-Angle Formulas

Formulas for Lowering Powers

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

EXAMPLE: Express $\sin^2 x \cos^2 x$ in terms of the first power of cosine.

Solution: We use the formulas for lowering powers repeatedly.

$$\begin{aligned}\sin^2 x \cos^2 x &= \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) = \frac{(1-\cos 2x)(1+\cos 2x)}{2\cdot 2}\\ &= \frac{1-\cos^2 2x}{4} = \frac{1}{4} - \frac{1}{4}\cos^2 2x = \frac{1}{4} - \frac{1}{4}\left(\frac{1+\cos 4x}{2}\right)\\ &= \frac{1}{4} - \frac{1+\cos 4x}{8} = \frac{1}{4} - \frac{1}{8} - \frac{\cos 4x}{8} = \frac{1}{8} - \frac{1}{8}\cos 4x = \frac{1}{8}(1-\cos 4x)\end{aligned}$$

Another way to obtain this identity is to use the double-angle formula for sine in the form $\sin x \cos x = \frac{1}{2}\sin 2x$. Thus

$$\sin^2 x \cos^2 x = \frac{1}{4}\sin^2 2x = \frac{1}{4}\left(\frac{1-\cos 4x}{2}\right) = \frac{1}{8}(1-\cos 4x)$$

Half-Angle Formulas

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$

The choice of the $+$ or $-$ sign depends on the quadrant in which $u/2$ lies.

EXAMPLE: Find the exact value of $\sin 22.5°$.

EXAMPLE: Find the exact value of $\sin 22.5^\circ$.

Solution: Since 22.5° is half of 45°, we use the half-angle formula for sine with $u = 45^\circ$. We choose the $+$ sign because 22.5° is in the first quadrant.

$$\sin\frac{45^\circ}{2} = \sqrt{\frac{1-\cos 45^\circ}{2}} = \sqrt{\frac{1-\sqrt{2}/2}{2}} = \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{1}{2}\sqrt{2-\sqrt{2}}$$

EXAMPLE: Find the exact value of $\sin 15^\circ$.

Solution: Since 15° is half of 30°, we use the half-angle formula for sine with $u = 30^\circ$. We choose the $+$ sign because 15° is in the first quadrant.

$$\sin\frac{30^\circ}{2} = \sqrt{\frac{1-\cos 30^\circ}{2}} = \sqrt{\frac{1-\sqrt{3}/2}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{1}{2}\sqrt{2-\sqrt{3}}$$

Note that $\frac{1}{2}\sqrt{2-\sqrt{3}}$ can also be rewritten as $\frac{\sqrt{6}-\sqrt{2}}{4}$. Indeed,

$$\frac{1}{2}\sqrt{2-\sqrt{3}} = \frac{\sqrt{6}-\sqrt{2}}{4}$$
$$\Uparrow$$
$$2\sqrt{2-\sqrt{3}} = \sqrt{6}-\sqrt{2}$$
$$\Uparrow$$
$$\left(2\sqrt{2-\sqrt{3}}\right)^2 = \left(\sqrt{6}-\sqrt{2}\right)^2$$
$$\Uparrow$$
$$4\left(2-\sqrt{3}\right) = \left(\sqrt{6}\right)^2 - 2\sqrt{6}\sqrt{2} + \left(\sqrt{2}\right)^2$$
$$\Uparrow$$
$$8-4\sqrt{3} = 6-2\sqrt{12}+2$$
$$\Uparrow$$
$$8-4\sqrt{3} = 8-2\sqrt{12}$$
$$\Uparrow$$
$$8-4\sqrt{3} = 8-2\sqrt{4}\sqrt{3}$$

EXAMPLE: Find $\tan(u/2)$ if $\sin u = \frac{2}{5}$ and u is in quadrant II.

Solution: To use the half-angle formulas for tangent, we first need to find $\cos u$. Since cosine is negative in quadrant II, we have

$$\cos u = -\sqrt{1-\sin^2 u} = -\sqrt{1-\left(\frac{2}{5}\right)^2} = -\frac{\sqrt{21}}{5}$$

Thus

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{1+\sqrt{21}/5}{2/5} = \frac{5+\sqrt{21}}{2}$$

Product-Sum Formulas

Product-to-Sum Formulas

$$\sin u \cos v = \tfrac{1}{2}[\sin(u + v) + \sin(u - v)]$$

$$\cos u \sin v = \tfrac{1}{2}[\sin(u + v) - \sin(u - v)]$$

$$\cos u \cos v = \tfrac{1}{2}[\cos(u + v) + \cos(u - v)]$$

$$\sin u \sin v = \tfrac{1}{2}[\cos(u - v) - \cos(u + v)]$$

EXAMPLE: Express $\sin 3x \sin 5x$ as a sum of trigonometric functions.

Solution: Using the fourth product-to-sum formula with $u = 3x$ and $v = 5x$ and the fact that cosine is an even function, we get

$$\begin{aligned}\sin 3x \sin 5x &= \frac{1}{2}[\cos(3x - 5x) - \cos(3x + 5x)]\\ &= \frac{1}{2}\cos(-2x) - \frac{1}{2}\cos 8x\\ &= \frac{1}{2}\cos 2x - \frac{1}{2}\cos 8x\end{aligned}$$

Sum-to-Product Formulas

$$\sin x + \sin y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2}$$

$$\sin x - \sin y = 2 \cos \frac{x + y}{2} \sin \frac{x - y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x + y}{2} \cos \frac{x - y}{2}$$

$$\cos x - \cos y = -2 \sin \frac{x + y}{2} \sin \frac{x - y}{2}$$

EXAMPLES:

(a) Write $\sin 7x + \sin 3x$ as a product.

(b) Verify the identity $\dfrac{\sin 3x - \sin x}{\cos 3x + \cos x} = \tan x$.

EXAMPLES:

(a) Write $\sin 7x + \sin 3x$ as a product.

Solution: The first sum-to-product formula gives

$$\sin 7x + \sin 3x = 2\sin\frac{7x+3x}{2}\cos\frac{7x-3x}{2} = 2\sin 5x\cos 2x$$

(b) Verify the identity $\dfrac{\sin 3x - \sin x}{\cos 3x + \cos x} = \tan x$.

Solution: We apply the second sum-to-product formula to the numerator and the third formula to the denominator.

$$\frac{\sin 3x - \sin x}{\cos 3x + \cos x} = \frac{2\cos\dfrac{3x+x}{2}\sin\dfrac{3x-x}{2}}{2\cos\dfrac{3x+x}{2}\cos\dfrac{3x-x}{2}} = \frac{2\cos 2x\sin x}{2\cos 2x\cos x} = \frac{\sin x}{\cos x} = \tan x$$

EXAMPLE: Verify the identity $\dfrac{\sin 4x + \sin 2x}{\sin 2x} = \dfrac{\sin 3x}{\sin x}$.

Solution: We apply the first sum-to-product formula to the numerator.

$$\frac{\sin 4x + \sin 2x}{\sin 2x} = \frac{2\sin\dfrac{4x+2x}{2}\cos\dfrac{4x-2x}{2}}{2\sin x\cos x} = \frac{2\sin 3x\cos x}{2\sin x\cos x} = \frac{\sin 3x}{\sin x}$$