Section 7.1 Trigonometric Identities

Fundamental Trigonometric Identities

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

Even-Odd Identities

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - u\right) = \cos u \qquad \tan\left(\frac{\pi}{2} - u\right) = \cot u \qquad \sec\left(\frac{\pi}{2} - u\right) = \csc u$$

$$\cos\left(\frac{\pi}{2} - u\right) = \sin u \qquad \cot\left(\frac{\pi}{2} - u\right) = \tan u \qquad \csc\left(\frac{\pi}{2} - u\right) = \sec u$$

EXAMPLE: Simplify the expression $\cos t + \tan t \sin t$.

Solution: We have

$$\cos t + \tan t \sin t = \cos t + \left(\frac{\sin t}{\cos t}\right)\sin t = \cos t + \frac{\sin^2 t}{\cos t} = \frac{\cos^2 t}{\cos t} + \frac{\sin^2 t}{\cos t}$$

$$= \frac{\cos^2 t + \sin^2 t}{\cos t} = \frac{1}{\cos t} = \sec t$$

EXAMPLE: Simplify the expression $\tan\theta + \dfrac{\cos\theta}{1 + \sin\theta}$.

Solution: We have

$$\tan\theta + \frac{\cos\theta}{1+\sin\theta} = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{1+\sin\theta} = \frac{\sin\theta(1+\sin\theta) + \cos^2\theta}{\cos\theta(1+\sin\theta)} = \frac{\sin\theta + \sin^2\theta + \cos^2\theta}{\cos\theta(1+\sin\theta)}$$

$$= \frac{\sin\theta + 1}{\cos\theta(1+\sin\theta)}$$

$$= \frac{1}{\cos\theta} = \sec\theta$$

Guidelines for Proving Trigonometric Identities

1. **Start with one side.** Pick one side of the equation and write it down. Your goal is to transform it into the other side. It's usually easier to start with the more complicated side.
2. **Use known identities.** Use algebra and the identities you know to change the side you started with. Bring fractional expressions to a common denominator, factor, and use the fundamental identities to simplify expressions.
3. **Convert to sines and cosines.** If you are stuck, you may find it helpful to rewrite all functions in terms of sines and cosines.

EXAMPLE: Verify the identity $\cos\theta(\sec\theta - \cos\theta) = \sin^2\theta$.

Solution: We have

$$\cos\theta(\sec\theta - \cos\theta) = \cos\theta\left(\frac{1}{\cos\theta} - \cos\theta\right) = 1 - \cos^2\theta = \sin^2\theta$$

EXAMPLE: Verify the identity $2\tan x\sec x = \dfrac{1}{1-\sin x} - \dfrac{1}{1+\sin x}$.

Solution: We have

$$\frac{1}{1-\sin x} - \frac{1}{1+\sin x} = \frac{(1+\sin x)-(1-\sin x)}{(1-\sin x)(1+\sin x)} = \frac{1+\sin x - 1 + \sin x}{(1-\sin x)(1+\sin x)}$$

$$= \frac{2\sin x}{1-\sin^2 x} = \frac{2\sin x}{\cos^2 x} = 2\frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}$$

$$= 2\tan x\sec x$$

EXAMPLE: Verify the identity $\dfrac{\cos u}{1-\sin u} = \sec u + \tan u$.

Solution: We have

$$\frac{\cos u}{1-\sin u} = \frac{\cos u(1+\sin u)}{(1-\sin u)(1+\sin u)} = \frac{\cos u(1+\sin u)}{1-\sin^2 u} = \frac{\cos u(1+\sin u)}{\cos^2 u}$$

$$= \frac{1+\sin u}{\cos u} = \frac{1}{\cos u} + \frac{\sin u}{\cos u}$$

$$= \sec u + \tan u$$

EXAMPLE: Verify the following identities:

(a) $\dfrac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = 1 - \sin x\cos x$

(b) $\dfrac{1+\cos u}{\cos u} = \dfrac{\tan^2 u}{\sec u - 1}$

EXAMPLE: Verify the following identities:

(a) $\dfrac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = 1 - \sin x \cos x$ (b) $\dfrac{1 + \cos u}{\cos u} = \dfrac{\tan^2 u}{\sec u - 1}$

Solution:

(a) We have

$$\begin{aligned}\frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} &= \frac{(\sin x + \cos x)(\sin^2 x - \sin x \cos x + \cos^2 x)}{\sin x + \cos x} \\ &= \sin^2 x - \sin x \cos x + \cos^2 x \\ &= 1 - \sin x \cos x\end{aligned}$$

(b) We have

$$\frac{1 + \cos u}{\cos u} = \frac{1}{\cos u} + \frac{\cos u}{\cos u} = \sec u + 1$$

and

$$\frac{\tan^2 u}{\sec u - 1} = \frac{\sec^2 u - 1}{\sec u - 1} = \frac{(\sec u - 1)(\sec u + 1)}{\sec u - 1} = \sec u + 1$$

so the equation is an identity.

We can also prove the identity in an other way:

$$\begin{aligned}\frac{\tan^2 u}{\sec u - 1} = \frac{\dfrac{\sin^2 u}{\cos^2 u}}{\dfrac{1}{\cos u} - 1} = \frac{\left(\dfrac{\sin^2 u}{\cos^2 u}\right)\cos^2 u}{\left(\dfrac{1}{\cos u} - 1\right)\cos^2 u} = \frac{\sin^2 u}{(1 - \cos u)\cos u} &= \frac{1 - \cos^2 u}{(1 - \cos u)\cos u} \\ &= \frac{(1 - \cos u)(1 + \cos u)}{(1 - \cos u)\cos u} \\ &= \frac{1 + \cos u}{\cos u}\end{aligned}$$

Finally, the fastest way to prove the identity is the following:

$$\begin{aligned}\frac{\tan^2 u}{\sec u - 1} = \frac{\tan^2 u(\sec u + 1)}{(\sec u - 1)(\sec u + 1)} = \frac{\tan^2 u(\sec u + 1)}{\sec^2 u - 1} = \frac{\tan^2 u(\sec u + 1)}{\tan^2 u} &= \sec u + 1 \\ &= \frac{1 + \cos u}{\cos u}\end{aligned}$$