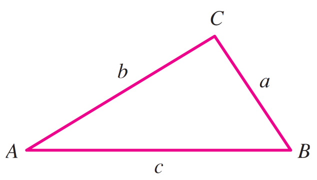
Section 6.6 The Law of Cosines

The Law of Sines cannot be used directly to solve triangles if we know two sides and the angle between them or if we know all three sides (these are Cases 3 and 4 of the preceding section). In these two cases, the **Law of Cosines** applies.



The Law of Cosines

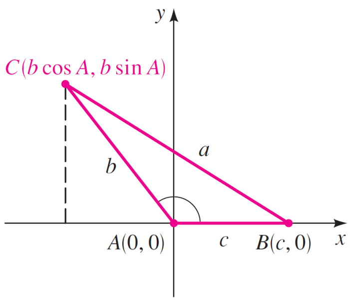
In any triangle ABC (see Figure 1), we have

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

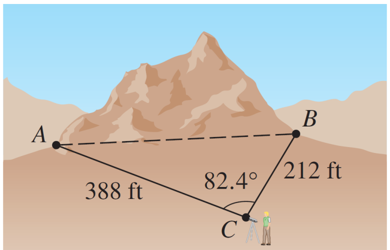
Proof: To prove the Law of Cosines, place triangle ABC so that $\angle A$ is at the origin, as shown in the Figure on the right. The coordinates of the vertices B and C are $(c, 0)$ and $(b\cos A, b\sin A)$, respectively. Using the Distance Formula, we get



$$\begin{aligned} a^2 &= (c - b\cos A)^2 + (b\sin A - 0)^2 \\ &= c^2 - 2bc\cos A + b^2\cos^2 A + b^2\sin^2 A \\ &= c^2 - 2bc\cos A + b^2(\cos^2 A + \sin^2 A) \\ &= b^2 + c^2 - 2bc\cos A \end{aligned}$$

This proves the first formula. The other two formulas are obtained in the same way by placing each of the other vertices of the triangle at the origin and repeating the preceding argument. ■

EXAMPLE: A tunnel is to be built through a mountain. To estimate the length of the tunnel, a surveyor makes the measurements shown in the Figure on the right. Use the surveyor's data to approximate the length of the tunnel.



Solution: To approximate the length c of the tunnel, we use the Law of Cosines:

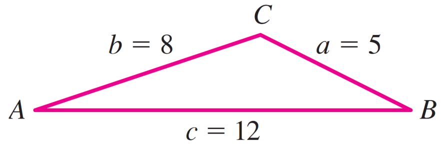
$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ &= 212^2 + 388^2 - 2(212)(388)\cos 82.4^\circ \\ &\approx 173730.2367 \end{aligned}$$

$$c \approx \sqrt{173730.2367} \approx 416.8$$

Thus, the tunnel will be approximately 417 ft long.

EXAMPLE: The sides of a triangle are $a = 5$, $b = 8$, and $c = 12$. Find the angles of the triangle.

EXAMPLE: The sides of a triangle are $a = 5, b = 8$, and $c = 12$. Find the angles of the triangle.



Solution: We first find $\angle A$. From the Law of Cosines, we have $a^2 = b^2 + c^2 - 2bc \cos A$. Solving for $\cos A$, we get

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{8^2 + 12^2 - 5^2}{2(8)(12)} = \frac{183}{192} = 0.953125$$

Using a calculator, we find that $\angle A = \cos^{-1}(0.953125) \approx 18^\circ$. In the same way the equations

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{5^2 + 12^2 - 8^2}{2(5)(12)} = 0.875$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{5^2 + 8^2 - 12^2}{2(5)(8)} = -0.6875$$

give $\angle B \approx 29^\circ$ and $\angle C \approx 133^\circ$. Of course, once two angles are calculated, the third can more easily be found from the fact that the sum of the angles of a triangle is 180°. However, it's a good idea to calculate all three angles using the Law of Cosines and add the three angles as a check on your computations.

EXAMPLE: Solve triangle ABC, where $\angle A = 46.5^\circ, b = 10.5$, and $c = 18.0$.

Solution: We can find a using the Law of Cosines.

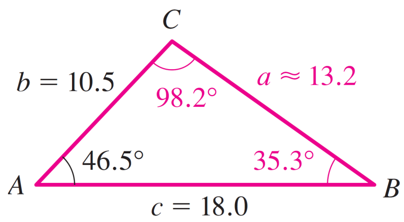
$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ &= (10.5)^2 + (18.0)^2 - 2(10.5)(18.0)(\cos 46.5^\circ) \approx 174.05 \end{aligned}$$

Thus, $a \approx \sqrt{174.05} \approx 13.2$. We also use the Law of Cosines to find $\angle B$ and $\angle C$, as in the previous Example.

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{13.2^2 + 18.0^2 - 10.5^2}{2(13.2)(18.0)} \approx 0.816477$$

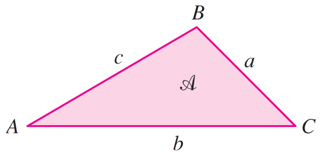
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{13.2^2 + 10.5^2 - 18.0^2}{2(13.2)(10.5)} \approx -0.142532$$

Using a calculator, we find that $\angle B \approx 35.3^\circ$ and $\angle C \approx 98.2^\circ$.



REMARK: We could have used the Law of Sines to find $\angle B$ and $\angle C$ in the Example above, since we knew all three sides and an angle in the triangle. But knowing the sine of an angle does not uniquely specify the angle, since an angle θ and its supplement $180^\circ - \theta$ both have the same sine. Thus we would need to decide which of the two angles is the correct choice. This ambiguity does not arise when we use the Law of Cosines, because every angle between 0° and 180° has a unique cosine. So using only the Law of Cosines is preferable in problems like the Example above.

The Area of a Triangle



Heron's Formula

The area $\mathscr{A}$ of triangle ABC is given by

$$\mathscr{A} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$ is the **semiperimeter** of the triangle; that is, s is half the perimeter.

Proof: We start with the formula $\mathcal{A} = \dfrac{1}{2}ab\sin C$ from Section 6.3. Thus

$$\begin{aligned}\mathcal{A}^2 &= \frac{1}{4}a^2b^2\sin^2 C\\ &= \frac{1}{4}a^2b^2(1-\cos^2 C)\\ &= \frac{1}{4}a^2b^2(1-\cos C)(1+\cos C)\end{aligned}$$

Next, we write the expressions $1-\cos C$ and $1+\cos C$ in terms of a, b, and c. By the Law of Cosines we have

$$\begin{aligned}\cos C &= \frac{a^2+b^2-c^2}{2ab}\\ 1+\cos C &= 1+\frac{a^2+b^2-c^2}{2ab}\\ &= \frac{2ab+a^2+b^2-c^2}{2ab}\\ &= \frac{(a+b)^2-c^2}{2ab}\\ &= \frac{(a+b+c)(a+b-c)}{2ab}\end{aligned}$$

Similarly

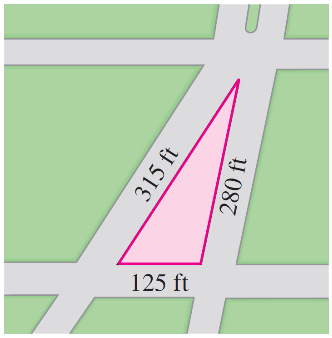
$$1-\cos C = \frac{(c+a-b)(c-a+b)}{2ab}$$

Substituting these expressions in the formula we obtained for $\mathcal{A}^2$ gives

$$\begin{aligned}\mathcal{A}^2 &= \frac{1}{4}a^2b^2\frac{(a+b+c)(a+b-c)}{2ab}\,\frac{(c+a-b)(c-a+b)}{2ab}\\ &= \frac{(a+b+c)}{2}\,\frac{(a+b-c)}{2}\,\frac{(c+a-b)}{2}\,\frac{(c-a+b)}{2}\\ &= s(s-c)(s-b)(s-a)\end{aligned}$$

Heron's Formula now follows by taking the square root of each side. ■

EXAMPLE: A businessman wishes to buy a triangular lot in a busy downtown location (see the Figure on the right). The lot frontages on the three adjacent streets are 125, 280, and 315 ft. Find the area of the lot.



Solution: The semiperimeter of the lot is

$$s = \frac{125 + 280 + 315}{2} = 360$$

By Heron's Formula the area is

$$\mathcal{A} = \sqrt{360(360 - 125)(360 - 280)(360 - 315)} \approx 17,451.6$$

Thus, the area is approximately 17,452 ft^2.