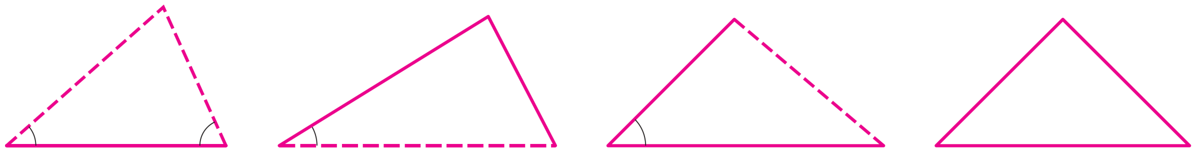
Section 6.5 The Law of Sines

To solve a triangle, we need to know certain information about its sides and angles. A triangle is determined by three of its six parts (angles and sides) as long as at least one of these three parts is a side.



(a) ASA or SAA (b) SSA (c) SAS (d) SSS

So, the possibilities, illustrated in the Figures above, are as follows.

Case 1 One side and two angles (ASA or SAA)

Case 2 Two sides and the angle opposite one of those sides (SSA)

Case 3 Two sides and the included angle (SAS)

Case 4 Three sides (SSS)

Cases 1 and 2 are solved using the Law of Sines; Cases 3 and 4 require the Law of Cosines.

The Law of Sines

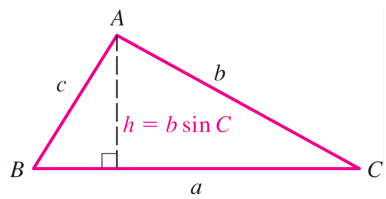
The **Law of Sines** says that in any triangle the lengths of the sides are proportional to the sines of the corresponding opposite angles.

The Law of Sines

In triangle ABC we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

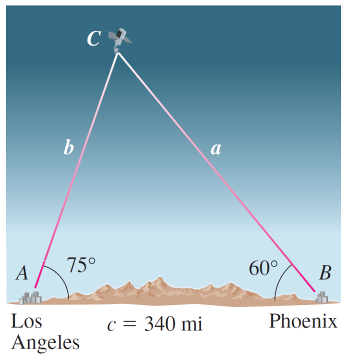
Proof: To see why the Law of Sines is true, refer to the Figure on the right. By the formula in Section 6.3 the area of triangle ABC is $\frac{1}{2}ab\sin C$. By the same formula the area of this triangle is also $\frac{1}{2}ac\sin B$ and $\frac{1}{2}bc\sin A$. Thus,



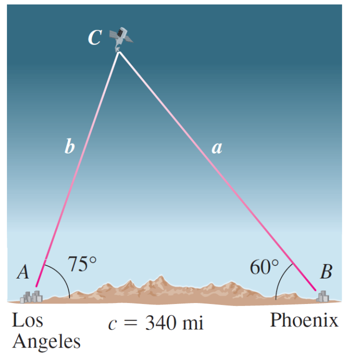
$$\frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C$$

Multiplying by $2/(abc)$ gives the Law of Sines. ■

EXAMPLE: A satellite orbiting the earth passes directly overhead at observation stations in Phoenix and Los Angeles, 340 mi apart. At an instant when the satellite is between these two stations, its angle of elevation is simultaneously observed to be 60° at Phoenix and 75° at Los Angeles. How far is the satellite from Los Angeles?



EXAMPLE: A satellite orbiting the earth passes directly overhead at observation stations in Phoenix and Los Angeles, 340 mi apart. At an instant when the satellite is between these two stations, its angle of elevation is simultaneously observed to be 60° at Phoenix and 75° at Los Angeles. How far is the satellite from Los Angeles?



Solution: Whenever two angles in a triangle are known, the third angle can be determined immediately because the sum of the angles of a triangle is 180°. In this case, $\angle C = 180^\circ - (75^\circ + 60^\circ) = 45^\circ$ (see the Figure on the right), so we have

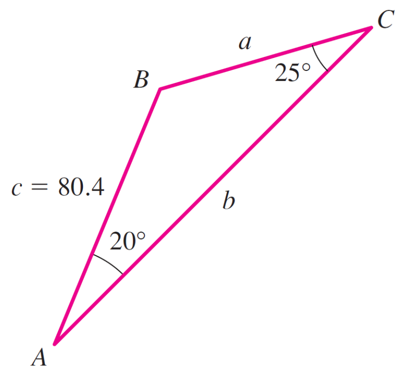
$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin 60^\circ}{b} = \frac{\sin 45^\circ}{340}$$

$$b = \frac{340 \sin 60^\circ}{\sin 45^\circ} = \frac{340 \cdot \frac{\sqrt{3}}{2}}{\frac{\sqrt{2}}{2}} = \left\{ \frac{340\sqrt{3}}{\sqrt{2}} = \frac{340\sqrt{3}\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{340\sqrt{6}}{2} \right\} = 170\sqrt{6} \approx 416.4$$

The distance of the satellite from Los Angeles is approximately 416.4 mi.

EXAMPLE: Solve the triangle in the Figure below.



Solution: Since side c is known, to find side a we use the relation

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

$$a = \frac{c \sin A}{\sin C} = \frac{80.4 \sin 20^\circ}{\sin 25^\circ} \approx 65.1$$

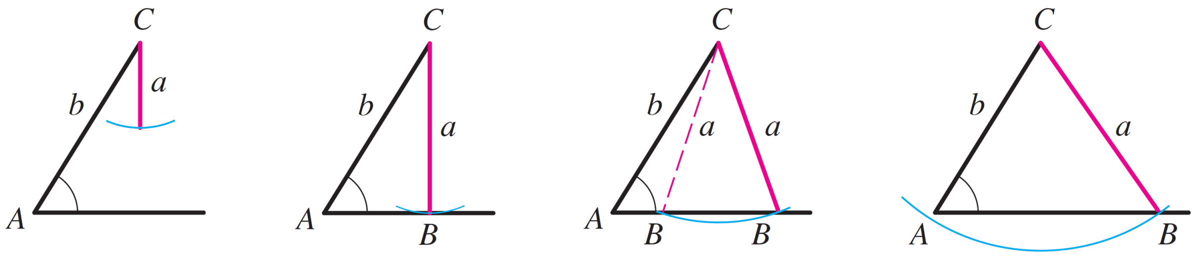
To find b, we first note that $\angle B = 180^\circ - (20^\circ + 25^\circ) = 135^\circ$. Therefore

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

$$b = \frac{c \sin B}{\sin C} = \frac{80.4 \sin 135^\circ}{\sin 25^\circ} \approx 134.5$$

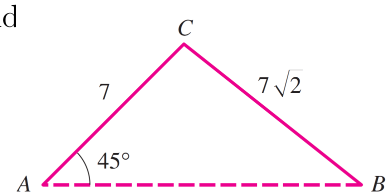
The Ambiguous Case

In the two previous Examples a unique triangle was determined by the information given. This is always true of Case 1 (ASA or SAA). But in Case 2 (SSA) there may be two triangles, one triangle, or no triangle with the given properties. For this reason, Case 2 is sometimes called the **ambiguous case**. To see why this is so, we show in the Figures below the possibilities when angle A and sides a and b are given. In part (a) no solution is possible, since side a is too short to complete the triangle. In part (b) the solution is a right triangle. In part (c) two solutions are possible, and in part (d) there is a unique triangle with the given properties.



(a) (b) (c) (d)

EXAMPLE: Solve triangle ABC, where $\angle A = 45°$, $a = 7\sqrt{2}$, and $b = 7$.



Solution: We first find $\angle B$.

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\sin B = \frac{b \sin A}{a} = \frac{7}{7\sqrt{2}} \sin 45° = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2}$$

So, $\sin B = \frac{1}{2}$, therefore B is either 30° or 150°. Since $\angle A = 45°$, we cannot have $\angle B = 150°$, because $45° + 150° > 180°$. Hence

$$\angle B = 30°$$

and the remaining angle is

$$\angle C = 180° - (30° + 45°) = 105°$$

Now we can find side c.

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

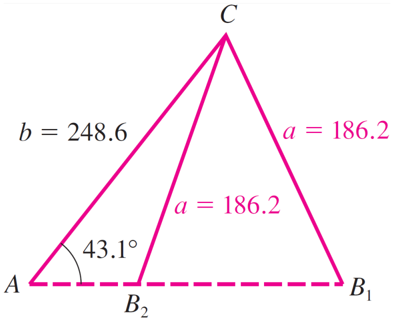
$$c = \frac{b \sin C}{\sin B} = \frac{7 \sin 105°}{\sin 30°} = \frac{7 \sin 105°}{1/2} = 14 \sin 105° \approx 13.5$$

EXAMPLE: Solve triangle ABC, where $\angle A = 43.1°$, $a = 186.2$, and $b = 248.6$.

EXAMPLE: Solve triangle ABC, where $\angle A = 43.1°$, $a = 186.2$, and $b = 248.6$.

Solution: From the given information we sketch the triangle shown in the Figure below. Note that side a may be drawn in two possible positions to complete the triangle. From the Law of Sines

$$\sin B = \frac{b \sin A}{a} = \frac{248.6 \sin 43.1°}{186.2} \approx 0.91225$$



There are two possible angles B between 0° and 180° such that $\sin B = 0.91225$. Using a calculator, we find that one of these angles is $\sin^{-1}(0.91225) \approx 65.8°$. The other is approximately $180° - 65.8° = 114.2°$. We denote these two angles by B_1 and B_2 so that

$$\angle B_1 \approx 65.8° \qquad \text{and} \qquad \angle B_2 \approx 114.2°$$

Thus two triangles satisfy the given conditions: triangle $A_1B_1C_1$ and triangle $A_2B_2C_2$.

Solve triangle $A_1B_1C_1$:

$$\angle C_1 \approx 180° - (43.1° + 65.8°) = 71.1°$$

Thus

$$c_1 = \frac{a_1 \sin C_1}{\sin A_1} \approx \frac{186.2 \sin 71.1°}{\sin 43.1°} \approx 257.8$$

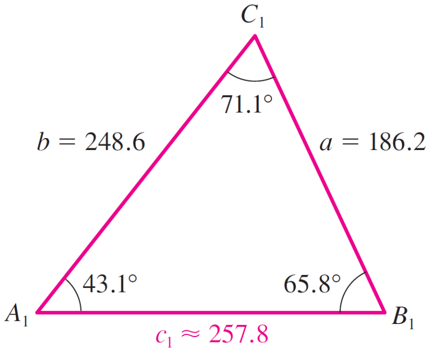
Solve triangle $A_2B_2C_2$:

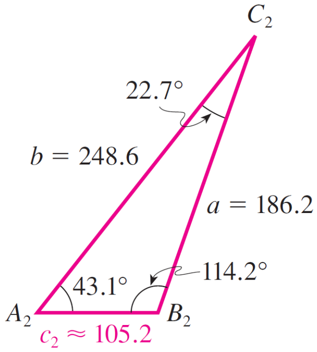
$$\angle C_2 \approx 180° - (43.1° + 114.2°) = 22.7°$$

Thus

$$c_2 = \frac{a_2 \sin C_2}{\sin A_2} \approx \frac{186.2 \sin 22.7°}{\sin 43.1°} \approx 105.2$$

Triangles $A_1B_1C_1$ and $A_2B_2C_2$ are shown in the Figures below.





EXAMPLE: Solve triangle ABC, where $\angle A = 42°$, $a = 70$, and $b = 122$.

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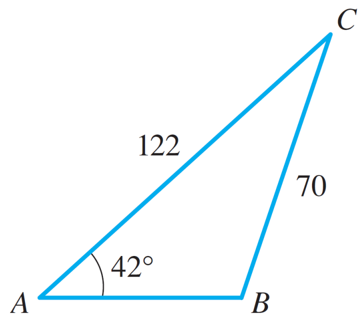

Solution: We have

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\sin B = \frac{b \sin A}{a} = \frac{122 \sin 42°}{70} \approx 1.17$$

Since the sine of an angle is never greater than 1, we conclude that no triangle satisfies the conditions given in this problem.