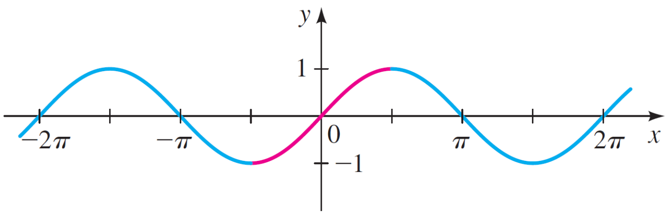
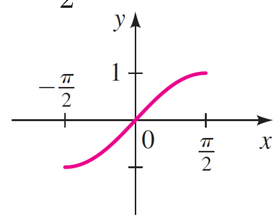
Section 6.4 Inverse Trigonometric Functions and Right Triangles

DEFINITION: The **inverse sine function,** denoted by $\sin^{-1} x$ (or $\arcsin x$), is defined to be the inverse of the restricted sine function

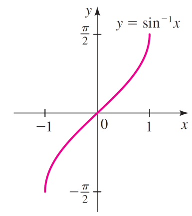
$$\sin x, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$



$y = \sin x$

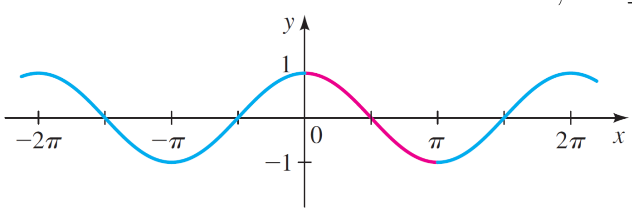


$y = \sin x,\ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

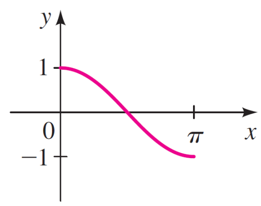


DEFINITION: The **inverse cosine function,** denoted by $\cos^{-1} x$ (or $\arccos x$), is defined to be the inverse of the restricted cosine function

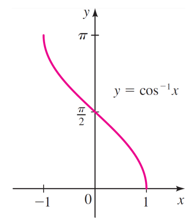
$$\cos x, \quad 0 \leq x \leq \pi$$



$y = \cos x$

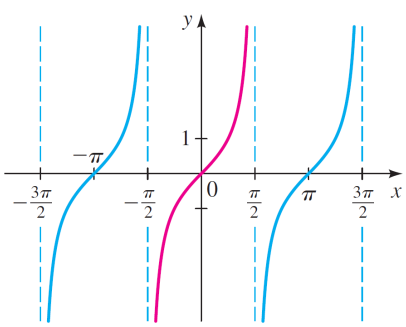


$y = \cos x,\ 0 \leq x \leq \pi$

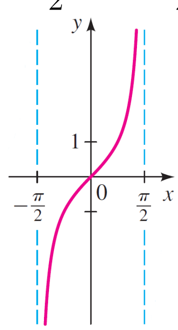


DEFINITION: The **inverse tangent function,** denoted by $\tan^{-1} x$ (or $\arctan x$), is defined to be the inverse of the restricted tangent function

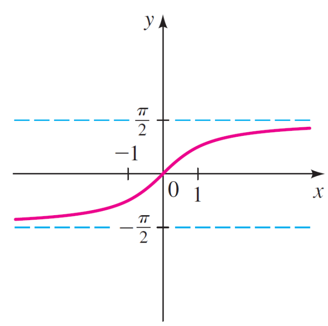
$$\tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$



$y = \tan x$



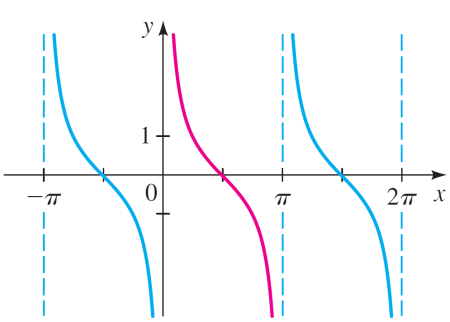
$y = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$



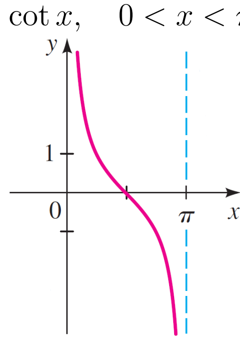
$y = \tan^{-1} x$

DEFINITION: The **inverse cotangent function,** denoted by $\cot^{-1} x$ (or $\operatorname{arccot} x$), is defined to be the inverse of the restricted cotangent function

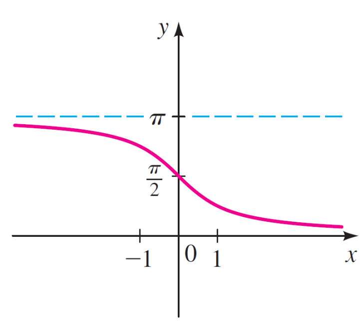
$$\cot x, \quad 0 < x < \pi$$



$y = \cot x$



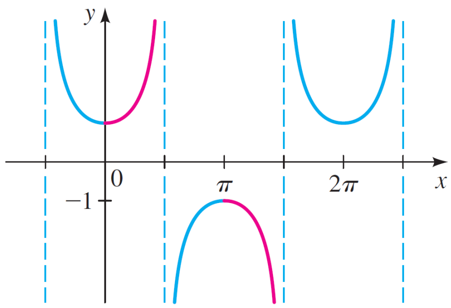
$y = \cot x,\ 0 < x < \pi$



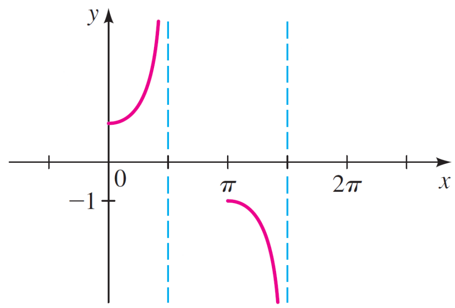
$y = \cot^{-1} x$

DEFINITION: The **inverse secant function,** denoted by $\sec^{-1} x$ (or $\operatorname{arcsec} x$), is defined to be the inverse of the restricted secant function

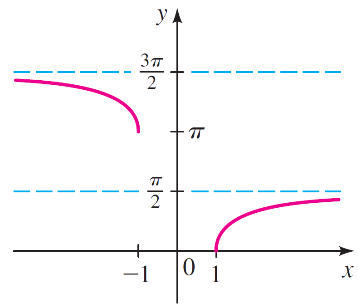
$$\sec x, \quad x \in [0, \pi/2) \cup [\pi, 3\pi/2) \quad \left[\text{or } x \in [0, \pi/2) \cup (\pi/2, \pi] \text{ in some other textbooks}\right]$$



$y = \sec x$



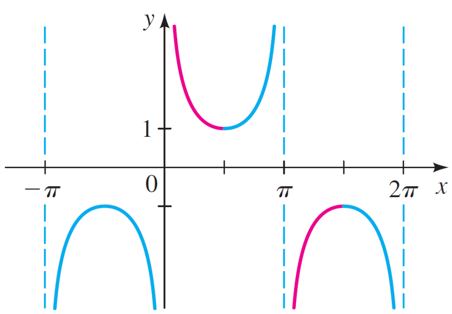
$y = \sec x,\ 0 \leq x < \frac{\pi}{2},\ \pi \leq x < \frac{3\pi}{2}$



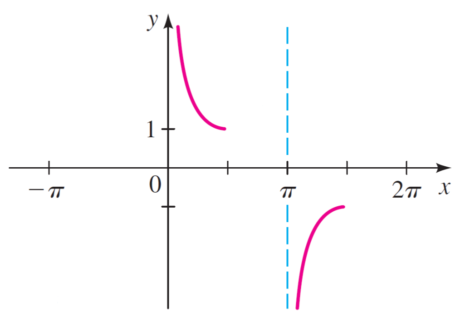
$y = \sec^{-1} x$

DEFINITION: The **inverse cosecant function,** denoted by $\csc^{-1} x$ (or $\operatorname{arccsc} x$), is defined to be the inverse of the restricted cosecant function

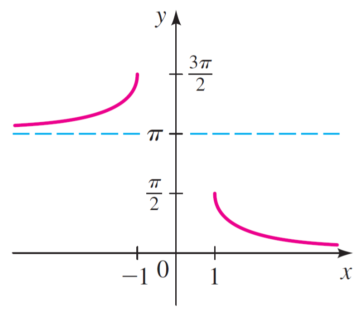
$$\csc x, \quad x \in (0, \pi/2] \cup (\pi, 3\pi/2] \quad \left[\text{or } x \in [-\pi/2, 0) \cup (0, \pi/2] \text{ in some other textbooks}\right]$$



$y = \csc x$



$y = \csc x,\ 0 < x \leq \frac{\pi}{2},\ \pi < x \leq \frac{3\pi}{2}$



$y = \csc^{-1} x$

<u>IMPORTANT:</u> Do not confuse

$$\sin^{-1} x, \quad \cos^{-1} x, \quad \tan^{-1} x, \quad \cot^{-1} x, \quad \sec^{-1} x, \quad \csc^{-1} x$$

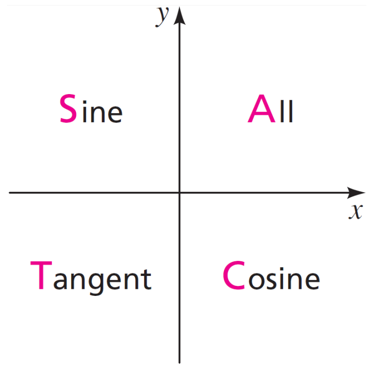
with

$$\frac{1}{\sin x}, \quad \frac{1}{\cos x}, \quad \frac{1}{\tan x}, \quad \frac{1}{\cot x}, \quad \frac{1}{\sec x}, \quad \frac{1}{\csc x}$$

FUNCTION	DOMAIN	RANGE
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1} x$	$(-\infty, +\infty)$	$(0, \pi)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup [\pi, 3\pi/2)$
$\csc^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$(0, \pi/2] \cup (\pi, 3\pi/2]$

FUNCTION	DOMAIN	RANGE
$\sin^{-1} x$	$[-1,1]$	$[-\pi/2,\pi/2]$
$\cos^{-1} x$	$[-1,1]$	$[0,\pi]$
$\tan^{-1} x$	$(-\infty,+\infty)$	$(-\pi/2,\pi/2)$
$\cot^{-1} x$	$(-\infty,+\infty)$	$(0,\pi)$
$\sec^{-1} x$	$(-\infty,-1]\cup[1,+\infty)$	$[0,\pi/2)\cup[\pi,3\pi/2)$
$\csc^{-1} x$	$(-\infty,-1]\cup[1,+\infty)$	$(0,\pi/2]\cup(\pi,3\pi/2]$

t	$\sin t$	$\cos t$	$\tan t$	$\csc t$	$\sec t$	$\cot t$
0	0	1	0	—	1	—
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	—	1	—	0



EXAMPLES:

(a) $\sin^{-1} 1 = \frac{\pi}{2}$, since $\sin\frac{\pi}{2} = 1$ and $\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(b) $\sin^{-1}(-1) = -\frac{\pi}{2}$, since $\sin\left(-\frac{\pi}{2}\right) = -1$ and $-\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(c) $\sin^{-1} 0 = 0$, since $\sin 0 = 0$ and $0 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(d) $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$, since $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(e) $\sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$, since $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(f) $\sin^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$, since $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

EXAMPLES:

$$\cos^{-1} 0 = \frac{\pi}{2}, \quad \cos^{-1} 1 = 0, \quad \cos^{-1}(-1) = \pi, \quad \cos^{-1}\frac{1}{2} = \frac{\pi}{3}, \quad \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}, \quad \cos^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$$

$$\tan^{-1} 1 = \frac{\pi}{4}, \quad \tan^{-1}(-1) = -\frac{\pi}{4}, \quad \tan^{-1}\sqrt{3} = \frac{\pi}{3}, \quad \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}, \quad \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

EXAMPLES: Find $\sec^{-1} 1$, $\sec^{-1}(-1)$, and $\sec^{-1}(-2)$.

Solution: We have

$$\sec^{-1} 1 = 0, \qquad \sec^{-1}(-1) = \pi, \qquad \sec^{-1}(-2) = \frac{4\pi}{3}$$

since

$$\sec 0 = 1, \qquad \sec\pi = -1, \qquad \sec\frac{4\pi}{3} = -2$$

and

$$0, \pi, \frac{4\pi}{3} \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$$

Note that $\sec\frac{2\pi}{3}$ is also -2, but $\sec^{-1}(-2) \neq \frac{2\pi}{3}$, since

$$\frac{2\pi}{3} \notin \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$$

EXAMPLES: Evaluate $\sin\left(\arcsin\dfrac{\pi}{7}\right)$, $\arcsin\left(\sin\dfrac{\pi}{7}\right)$, and $\arcsin\left(\sin\dfrac{8\pi}{7}\right)$.

Solution: Since $\arcsin x$ is the inverse of the restricted sine function, we have

$$\sin(\arcsin x) = x \text{ if } x \in [-1,1] \qquad \text{and} \qquad \arcsin(\sin x) = x \text{ if } x \in [-\pi/2, \pi/2]$$

Therefore

$$\sin\left(\arcsin\frac{\pi}{7}\right) = \frac{\pi}{7} \qquad \text{and} \qquad \arcsin\left(\sin\frac{\pi}{7}\right) = \frac{\pi}{7}$$

but

$$\arcsin\left(\sin\frac{8\pi}{7}\right) = \arcsin\left(\sin\left(\frac{\pi}{7}+\pi\right)\right) = \arcsin\left(-\sin\frac{\pi}{7}\right) = -\arcsin\left(\sin\frac{\pi}{7}\right) = -\frac{\pi}{7}$$

EXAMPLES: Evaluate $\cot\left(\arcsin\dfrac{2}{5}\right)$ and $\sec\left(\arcsin\dfrac{2}{5}\right)$.

Solution 1: We have

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\pm\sqrt{1-\sin^2\theta}}{\sin\theta} \qquad \text{and} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\pm\sqrt{1-\sin^2\theta}}$$

Since $-\dfrac{\pi}{2} \le \arcsin x \le \dfrac{\pi}{2}$, it follows that $\cos(\arcsin x) \ge 0$. Therefore if $\theta = \arcsin\dfrac{2}{5}$, then

$$\cot\theta = \frac{\sqrt{1-\sin^2\theta}}{\sin\theta} \qquad \text{and} \qquad \sec\theta = \frac{1}{\sqrt{1-\sin^2\theta}}$$

hence

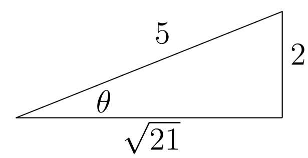
$$\cot\left(\arcsin\frac{2}{5}\right) = \frac{\sqrt{1-\sin^2\left(\arcsin\frac{2}{5}\right)}}{\sin\left(\arcsin\frac{2}{5}\right)} = \frac{\sqrt{1-\left(\frac{2}{5}\right)^2}}{\frac{2}{5}} = \frac{\sqrt{21}}{2}$$

and

$$\sec\left(\arcsin\frac{2}{5}\right) = \frac{1}{\sqrt{1-\sin^2\left(\arcsin\frac{2}{5}\right)}} = \frac{1}{\sqrt{1-\left(\frac{2}{5}\right)^2}} = \frac{5}{\sqrt{21}}$$

Solution 2: Put $\theta = \arcsin\dfrac{2}{5}$, so $\sin\theta = \dfrac{2}{5}$. Then

$$\cot\left(\arcsin\frac{2}{5}\right) = \cot\theta = \frac{\sqrt{21}}{2} \qquad \text{and} \qquad \sec\left(\arcsin\frac{2}{5}\right) = \sec\theta = \frac{5}{\sqrt{21}}$$



EXAMPLES: Evaluate, if possible, $\cot\left(\sin^{-1} 2\right)$ and $\sin\left(\tan^{-1} 2\right)$.

EXAMPLES: Evaluate, if possible, $\cot\left(\sin^{-1} 2\right)$ and $\sin\left(\tan^{-1} 2\right)$.

We first note that $\sin^{-1} 2$ does not exist, since $2 \notin [-1,1]$, that is, 2 is not in the domain of $\sin^{-1} x$. Therefore $\cot\left(\sin^{-1} 2\right)$ does not exist.

We will evaluate $\sin\left(\tan^{-1} 2\right)$ in two different ways:

Solution 1: We have

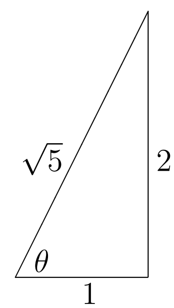
$$\sin\theta = \pm\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$

Since $-\pi/2 < \tan^{-1} x < \pi/2$, it follows that $\cos\left(\tan^{-1} x\right) > 0$. Therefore if $\theta = \tan^{-1} 2$, then

$$\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$

hence

$$\sin\left(\tan^{-1} 2\right) = \frac{\tan\left(\tan^{-1} 2\right)}{\sqrt{1+\tan^2\left(\tan^{-1} 2\right)}} = \frac{2}{\sqrt{1+2^2}} = \frac{2}{\sqrt{5}}$$



Solution 2: Put $\theta = \tan^{-1} 2 = \tan^{-1}\frac{2}{1}$, so $\tan\theta = \frac{2}{1}$. Then

$$\sin\left(\tan^{-1} 2\right) = \sin\theta = \frac{2}{\sqrt{5}}$$

EXAMPLES: Evaluate $\sin\left(\cot^{-1}\left(-\frac{1}{2}\right)\right)$ and $\cos\left(\cot^{-1}\left(-\frac{1}{2}\right)\right)$.

EXAMPLES: Evaluate $\sin\left(\cot^{-1}\left(-\frac{1}{2}\right)\right)$ and $\cos\left(\cot^{-1}\left(-\frac{1}{2}\right)\right)$.

Solution 1: We have

$$\sin\theta = \pm\frac{1}{\sqrt{1+\cot^2\theta}} \quad \text{and} \quad \cos\theta = \pm\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$$

Since $0 < \cot^{-1} x < \pi$, it follows that $\sin(\cot^{-1} x) > 0$. Therefore if $\theta = \cot^{-1}\left(-\frac{1}{2}\right)$, then

$$\sin\theta = \frac{1}{\sqrt{1+\cot^2\theta}} \quad \text{and} \quad \cos\theta = \frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$$

hence

$$\sin\left(\cot^{-1}\left(-\frac{1}{2}\right)\right) = \frac{1}{\sqrt{1+\cot^2\left(\cot^{-1}\left(-\frac{1}{2}\right)\right)}} = \frac{1}{\sqrt{1+\left(-\frac{1}{2}\right)^2}} = \frac{2}{\sqrt{5}}$$

and

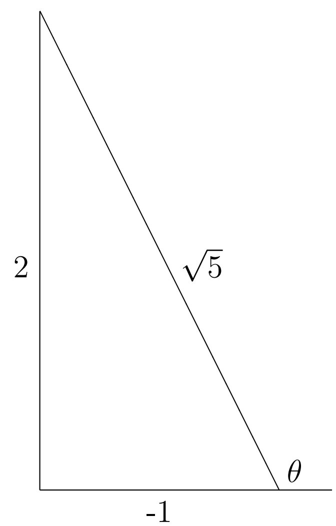
$$\cos\left(\cot^{-1}\left(-\frac{1}{2}\right)\right) = \frac{\cot\left(\cot^{-1}\left(-\frac{1}{2}\right)\right)}{\sqrt{1+\cot^2\left(\cot^{-1}\left(-\frac{1}{2}\right)\right)}} = \frac{-\frac{1}{2}}{\sqrt{1+\left(-\frac{1}{2}\right)^2}} = -\frac{1}{\sqrt{5}}$$

Solution 2: Put $\theta = \cot^{-1}\left(-\frac{1}{2}\right)$, so $\cot\theta = -\frac{1}{2} = \frac{-1}{2}$. Then

$$\sin\left(\cot^{-1}\left(-\frac{1}{2}\right)\right) = \sin\theta = \frac{2}{\sqrt{5}}$$

and

$$\cos\left(\cot^{-1}\left(-\frac{1}{2}\right)\right) = \cos\theta = -\frac{1}{\sqrt{5}}$$



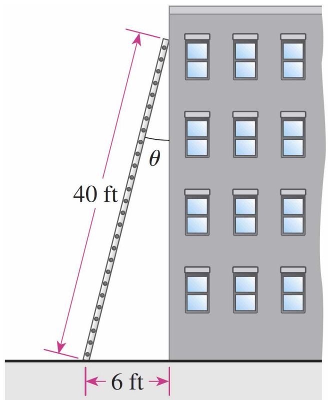
EXAMPLE: A 40-ft ladder leans against a building. If the base of the ladder is 6 ft from the base of the building, what is the angle formed by the ladder and the building?

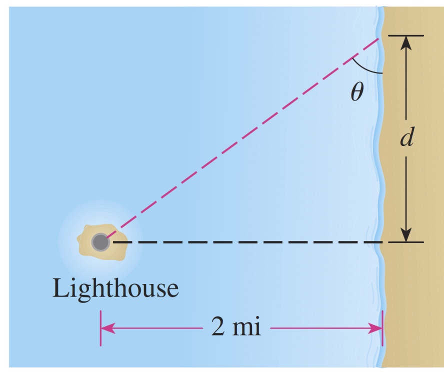
Solution: First we sketch a diagram as in the Figure below (left). If θ is the angle between the ladder and the building, then

$$\sin\theta = \frac{6}{40} = 0.15$$

Now we use $\sin^{-1}$ to find θ:

$$\theta = \sin^{-1}(0.15) \approx 8.6^\circ$$





EXAMPLE: A lighthouse is located on an island that is 2 mi off a straight shoreline (see the Figure above on the right). Express the angle formed by the beam of light and the shoreline in terms of the distance d in the figure.

Solution: From the figure we see that

$$\tan\theta = \frac{2}{d}$$

Taking the inverse tangent of both sides, we get

$$\tan^{-1}(\tan\theta) = \tan^{-1}\left(\frac{2}{d}\right)$$

$$\theta = \tan^{-1}\left(\frac{2}{d}\right)$$