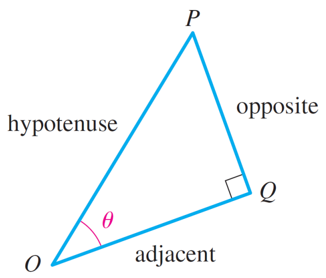
Section 6.3 Trigonometric Functions of Angles

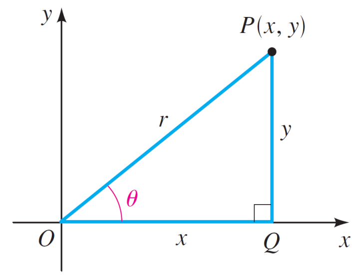
In the preceding section we defined the trigonometric ratios for acute angles. Here we extend the trigonometric ratios to all angles by defining the trigonometric functions of angles.

Trigonometric Functions of Angles

Let POQ be a right triangle with acute angle θ as shown in Figure (a) below. Place θ in standard position as shown in Figure (b) below.



(a)

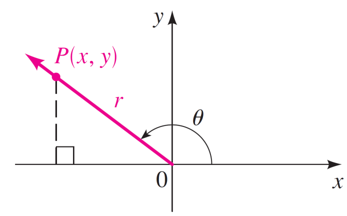


(b)

Then $P = P(x, y)$ is a point on the terminal side of θ. In triangle POQ, the opposite side has length y and the adjacent side has length x. Using the Pythagorean Theorem, we see that the hypotenuse has length $r = \sqrt{x^2 + y^2}$. So

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$



These observations allow us to extend the trigonometric ratios to any angle. We define the trigonometric functions of angles as follows

Definition of the Trigonometric Functions

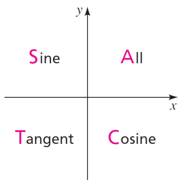
Let θ be an angle in standard position and let $P(x, y)$ be a point on the terminal side. If $r = \sqrt{x^2 + y^2}$ is the distance from the origin to the point $P(x, y)$, then

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

$$\csc\theta = \frac{r}{y} \quad (y \neq 0) \qquad \sec\theta = \frac{r}{x} \quad (x \neq 0) \qquad \cot\theta = \frac{x}{y} \quad (y \neq 0)$$

Evaluating Trigonometric Functions at Any Angle

The following mnemonic device will help you remember which trigonometric functions are positive in each quadrant: **A**ll of them, **S**ine, **T**angent, or **C**osine.



You can remember this as "**A**ll **S**tudents **T**ake **C**alculus."

Signs of the Trigonometric Functions

Quadrant	Positive functions	Negative functions
I	all	none
II	sin, csc	cos, sec, tan, cot
III	tan, cot	sin, csc, cos, sec
IV	cos, sec	sin, csc, tan, cot

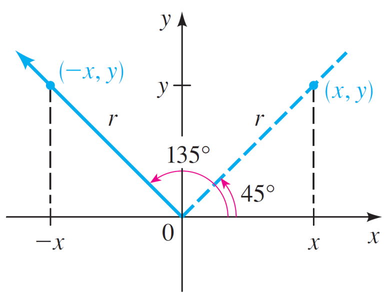
EXAMPLE: Find

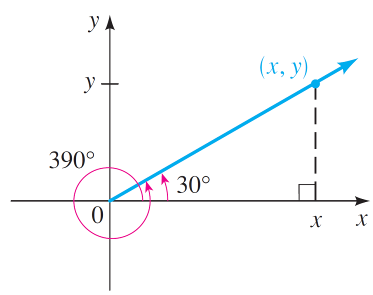
(a) $\cos 135^\circ$ (b) $\tan 390^\circ$

Solution:

(a) From the first Figure below we see that $\cos 135^\circ = -x/r$. But $\cos 45^\circ = x/r$, and since $\cos 45^\circ = \sqrt{2}/2$, we have $\cos 135^\circ = -\dfrac{\sqrt{2}}{2}$.

(b) The angles 390° and 30° are coterminal. From the second Figure below it's clear that $\tan 390^\circ = \tan 30^\circ$, and since $\tan 30^\circ = \sqrt{3}/3$, we have $\tan 390^\circ = \dfrac{\sqrt{3}}{3}$.

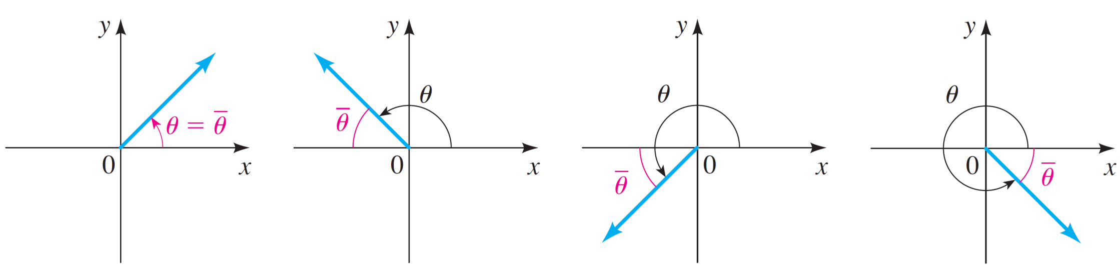




From the Example above we see that the trigonometric functions for angles that aren't acute have the same value, except possibly for sign, as the corresponding trigonometric functions of an acute angle. That acute angle will be called the *reference angle*.

Reference Angle

Let θ be an angle in standard position. The **reference angle** $\bar{\theta}$ associated with θ is the acute angle formed by the terminal side of θ and the x-axis.



EXAMPLE: Find the reference angle for

(a) $\theta = \dfrac{5\pi}{3}$ (b) $\theta = 870^\circ$

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Solution:

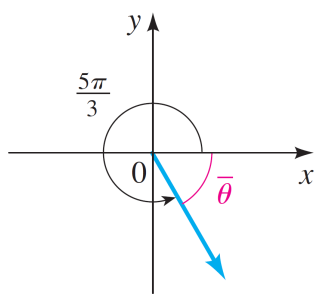
(a) Since

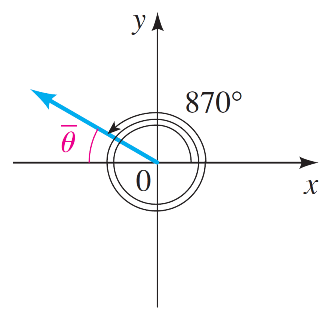
$$\frac{5\pi}{3} = \frac{6\pi - \pi}{3} = \frac{6\pi}{3} - \frac{\pi}{3} = 2\pi - \frac{\pi}{3}$$

the reference number for $5\pi/3$ is $\pi/3$ and the terminal point of $5\pi/3$ is in Quadrant IV.

(b) The angles 870° and 150° are coterminal [because $870 - 2(360) = 150$]. Thus, the terminal side of this angle is in Quadrant II (see the second Figure below). So the reference angle is

$$\overline{\theta} = 180^\circ - 150^\circ = 30^\circ$$





EXAMPLE: Find the reference angle for

(a) $\theta = \dfrac{17\pi}{6}$ (b) $\theta = \dfrac{13\pi}{7}$

Solution:

(a) We have $\dfrac{17\pi}{6} = \dfrac{18\pi - \pi}{6} = \dfrac{18\pi}{6} - \dfrac{\pi}{6} = 3\pi - \dfrac{\pi}{6}$, therefore $\overline{\theta} = \dfrac{\pi}{6}$ (Quadrant II).

(b) We have $\dfrac{13\pi}{7} = \dfrac{14\pi - \pi}{7} = \dfrac{14\pi}{7} - \dfrac{\pi}{7} = 2\pi - \dfrac{\pi}{7}$, therefore $\overline{\theta} = \dfrac{\pi}{7}$ (Quadrant IV).

Evaluating Trigonometric Functions for Any Angle

To find the values of the trigonometric functions for any angle θ, we carry out the following steps.

1. Find the reference angle $\overline{\theta}$ associated with the angle θ.
2. Determine the sign of the trigonometric function of θ by noting the quadrant in which θ lies.
3. The value of the trigonometric function of θ is the same, except possibly for sign, as the value of the trigonometric function of $\overline{\theta}$.

EXAMPLE: Find

(a) $\sin 240^\circ$ (b) $\cot 495^\circ$

EXAMPLE: Find

(a) $\sin 240°$ (b) $\cot 495°$

Solution:

(a) This angle has its terminal side in Quadrant III, as shown in the first Figure below. The reference angle is therefore $240° - 180° = 60°$, and the value of $\sin 240°$ is negative. Thus

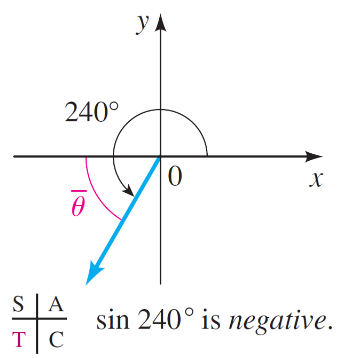
$$\sin 240° = -\sin 60° = -\frac{\sqrt{3}}{2}$$

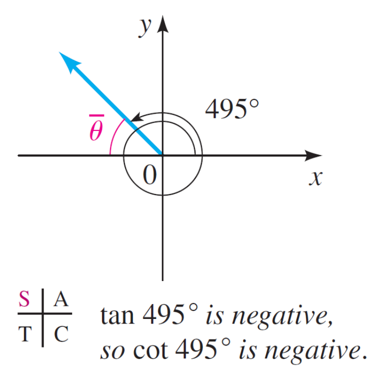
Sign Reference angle

(b) The angle 495° is coterminal with the angle 135° (since $495° - 360° = 135°$), and the terminal side of this angle is in Quadrant II, as shown in the second Figure below. So the reference angle is $180° - 135° = 45°$, and the value of $\cot 495°$ is negative. We have

$$\cot 495° = \cot 135° = -\cot 45° = -1$$

Coterminal angles Sign Reference angle





EXAMPLE: Find

(a) $\sin \dfrac{16\pi}{3}$ (b) $\sec\left(-\dfrac{\pi}{4}\right)$

EXAMPLE: Find

(a) $\sin\dfrac{16\pi}{3}$ (b) $\sec\left(-\dfrac{\pi}{4}\right)$

Solution:

(a) Since

$$\frac{16\pi}{3}=\frac{15\pi+\pi}{3}=\frac{15\pi}{3}+\frac{\pi}{3}=5\pi+\frac{\pi}{3}$$

the reference number for $16\pi/3$ is $\pi/3$ (see the first Figure below) and the terminal point of $16\pi/3$ is in Quadrant III. Thus $\sin(16\pi/3)$ is negative and

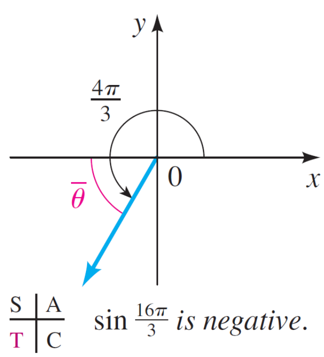
$$\sin\frac{16\pi}{3}=-\sin\frac{\pi}{3}=-\frac{\sqrt{3}}{2}$$

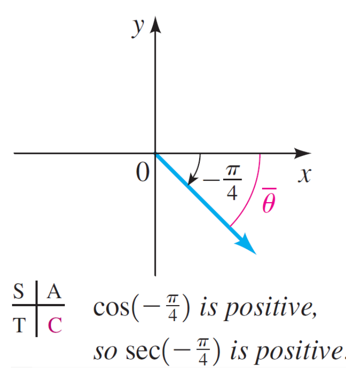
Sign Reference angle

(b) The angle $-\pi/4$ is in Quadrant IV, and its reference angle is $\pi/4$ (see the second Figure below). Since secant is positive in this quadrant, we get

$$\sec\left(-\frac{\pi}{4}\right)=+\sec\frac{\pi}{4}=\sqrt{2}$$

Sign Reference angle





Trigonometric Identities

Fundamental Identities

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

EXAMPLE:

(a) Express $\sin\theta$ in terms of $\cos\theta$.

(b) Express $\tan\theta$ in terms of $\sin\theta$ where θ is in Quadrant II.

Solution:

(a) From the first Pythagorean identity we get

$$\sin\theta = \pm\sqrt{1-\cos^2\theta}$$

where the sign depends on the quadrant. If θ is in Quadrant I or II, then $\sin\theta$ is positive, and hence

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

whereas if θ is in Quadrant III or IV, $\sin\theta$ is negative, and so

$$\sin\theta = -\sqrt{1-\cos^2\theta}$$

(b) Since $\tan\theta = \sin\theta/\cos\theta$, we need to write $\cos\theta$ in terms of $\sin\theta$. By part (a)

$$\cos\theta = \pm\sqrt{1-\sin^2\theta}$$

and since $\cos\theta$ is negative in Quadrant II, the negative sign applies here. Thus

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{-\sqrt{1-\sin^2\theta}}$$

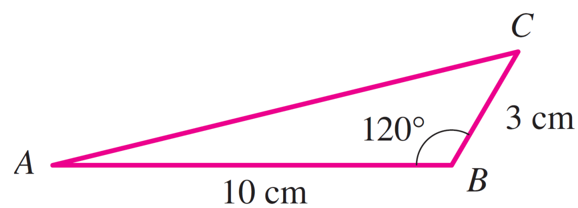
Areas of Triangles

Area of a Triangle

The area $\mathscr{A}$ of a triangle with sides of lengths a and b and with included angle θ is

$$\mathscr{A} = \tfrac{1}{2}ab\sin\theta$$

EXAMPLE: Find the area of triangle ABC shown in the Figure below.



Solution: The triangle has sides of length 10 cm and 3 cm, with included angle 120°. Therefore

$$\begin{aligned}\mathscr{A} &= \frac{1}{2}ab\sin\theta\\ &= \frac{1}{2}(10)(3)\sin 120^\circ\\ &= 15\sin 60^\circ\\ &= 15\frac{\sqrt{3}}{2} \approx 13 \text{ cm}^2\end{aligned}$$