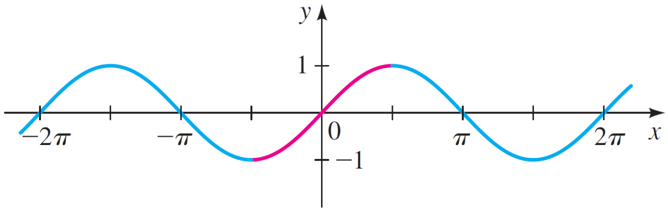
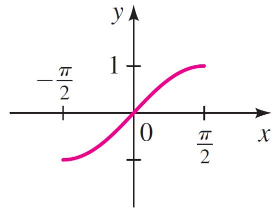
Section 5.5 Inverse Trigonometric Functions and Their Graphs

DEFINITION: The **inverse sine function,** denoted by $\sin^{-1} x$ (or $\arcsin x$), is defined to be the inverse of the restricted sine function

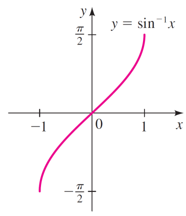
$$\sin x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$



$y = \sin x$

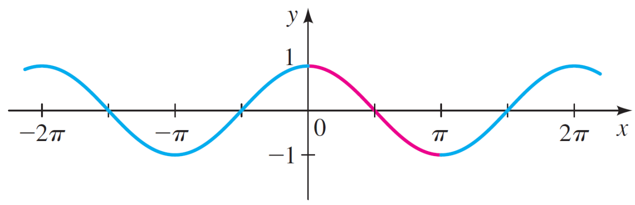


$y = \sin x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

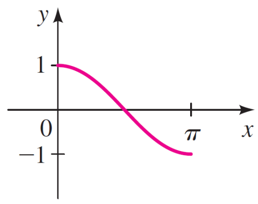


DEFINITION: The **inverse cosine function,** denoted by $\cos^{-1} x$ (or $\arccos x$), is defined to be the inverse of the restricted cosine function

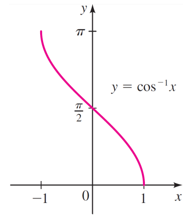
$$\cos x, \quad 0 \le x \le \pi$$



$y = \cos x$

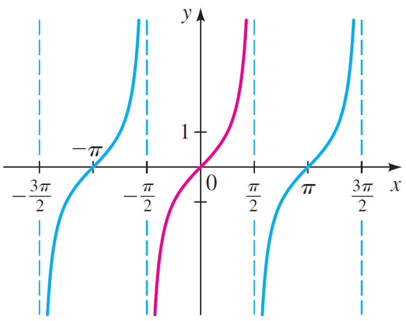


$y = \cos x,\ 0 \le x \le \pi$

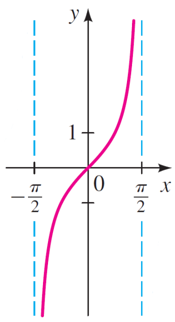


DEFINITION: The **inverse tangent function,** denoted by $\tan^{-1} x$ (or $\arctan x$), is defined to be the inverse of the restricted tangent function

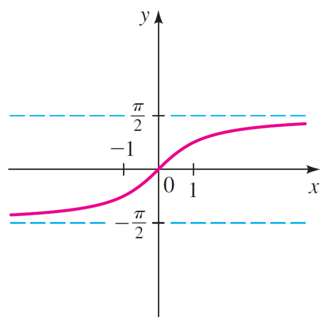
$$\tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$



$y = \tan x$



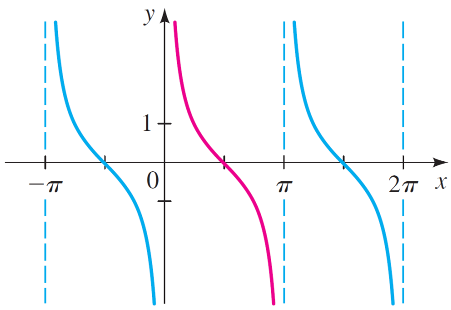
$y = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$



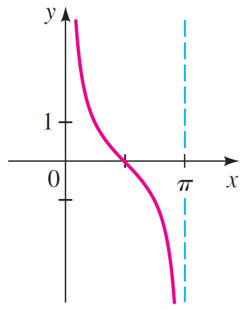
$y = \tan^{-1} x$

DEFINITION: The **inverse cotangent function,** denoted by $\cot^{-1} x$ (or $\operatorname{arccot} x$), is defined to be the inverse of the restricted cotangent function

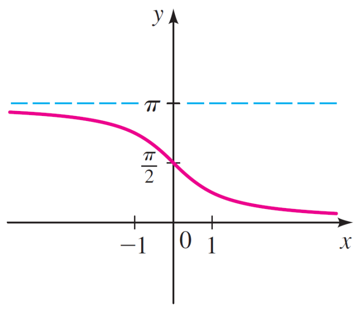
$$\cot x, \quad 0 < x < \pi$$



$y = \cot x$



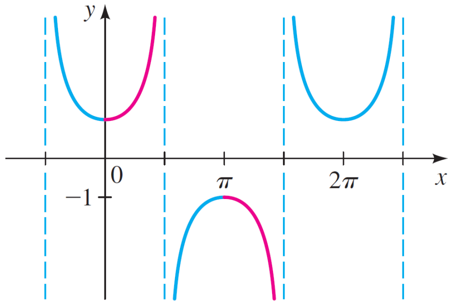
$y = \cot x,\ 0 < x < \pi$



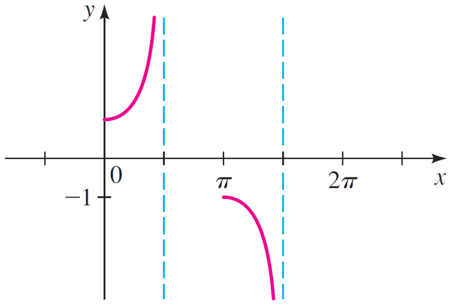
$y = \cot^{-1} x$

DEFINITION: The **inverse secant function,** denoted by $\sec^{-1} x$ (or $\operatorname{arcsec} x$), is defined to be the inverse of the restricted secant function

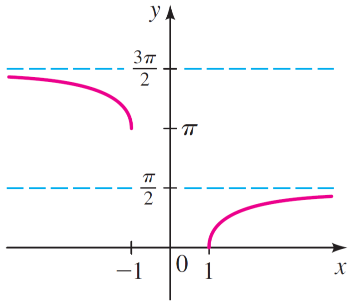
$$\sec x, \quad x \in [0, \pi/2) \cup [\pi, 3\pi/2) \quad \big[\text{or } x \in [0, \pi/2) \cup (\pi/2, \pi] \text{ in some other textbooks}\big]$$



$y = \sec x$



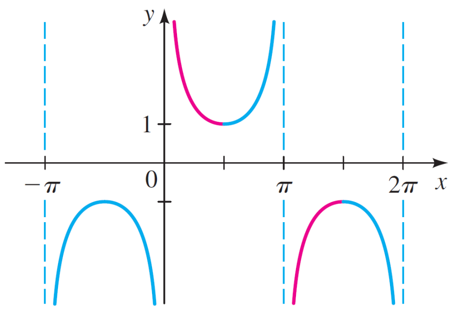
$y = \sec x,\ 0 \le x < \frac{\pi}{2},\ \pi \le x < \frac{3\pi}{2}$



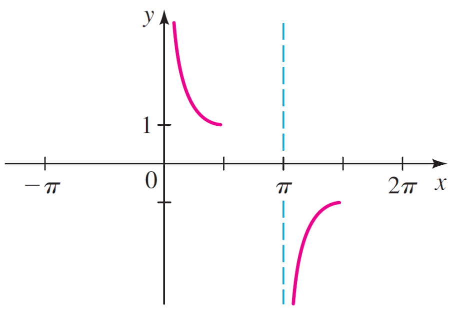
$y = \sec^{-1} x$

DEFINITION: The **inverse cosecant function,** denoted by $\csc^{-1} x$ (or $\operatorname{arccsc} x$), is defined to be the inverse of the restricted cosecant function

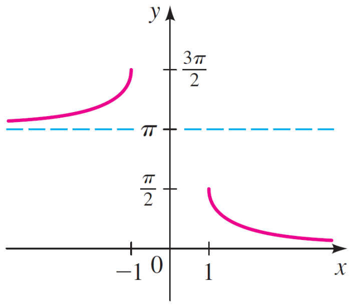
$$\csc x, \quad x \in (0, \pi/2] \cup (\pi, 3\pi/2] \quad \big[\text{or } x \in [-\pi/2, 0) \cup (0, \pi/2] \text{ in some other textbooks}\big]$$



$y = \csc x$



$y = \csc x,\ 0 < x \le \frac{\pi}{2},\ \pi < x \le \frac{3\pi}{2}$



$y = \csc^{-1} x$

<u>IMPORTANT:</u> Do not confuse

$$\sin^{-1} x, \quad \cos^{-1} x, \quad \tan^{-1} x, \quad \cot^{-1} x, \quad \sec^{-1} x, \quad \csc^{-1} x$$

with

$$\frac{1}{\sin x}, \quad \frac{1}{\cos x}, \quad \frac{1}{\tan x}, \quad \frac{1}{\cot x}, \quad \frac{1}{\sec x}, \quad \frac{1}{\csc x}$$

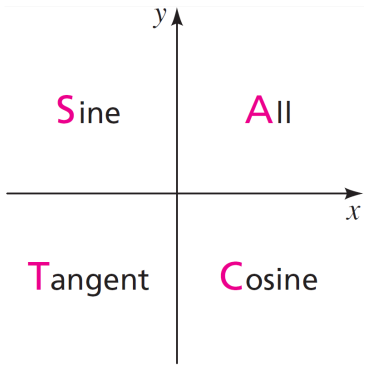
FUNCTION	DOMAIN	RANGE
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1} x$	$(-\infty, +\infty)$	$(0, \pi)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup [\pi, 3\pi/2)$
$\csc^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$(0, \pi/2] \cup (\pi, 3\pi/2]$

FUNCTION	DOMAIN	RANGE
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1} x$	$(-\infty, +\infty)$	$(0, \pi)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup [\pi, 3\pi/2)$
$\csc^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$(0, \pi/2] \cup (\pi, 3\pi/2]$

t	$\sin t$	$\cos t$	$\tan t$	$\csc t$	$\sec t$	$\cot t$
0	0	1	0	—	1	—
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	—	1	—	0

EXAMPLES:

(a) $\sin^{-1} 1 = \frac{\pi}{2}$, since $\sin \frac{\pi}{2} = 1$ and $\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



(b) $\sin^{-1}(-1) = -\frac{\pi}{2}$, since $\sin\left(-\frac{\pi}{2}\right) = -1$ and $-\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(c) $\sin^{-1} 0 = 0$, since $\sin 0 = 0$ and $0 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(d) $\sin^{-1} \frac{1}{2} = \frac{\pi}{6}$, since $\sin \frac{\pi}{6} = \frac{1}{2}$ and $\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(e) $\sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3}$, since $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(f) $\sin^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4}$, since $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

EXAMPLES:

$$\cos^{-1} 0 = \frac{\pi}{2}, \quad \cos^{-1} 1 = 0, \quad \cos^{-1}(-1) = \pi, \quad \cos^{-1} \frac{1}{2} = \frac{\pi}{3}, \quad \cos^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{6}, \quad \cos^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4}$$

$$\tan^{-1} 1 = \frac{\pi}{4}, \quad \tan^{-1}(-1) = -\frac{\pi}{4}, \quad \tan^{-1} \sqrt{3} = \frac{\pi}{3}, \quad \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}, \quad \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

EXAMPLES: Find $\sec^{-1} 1$, $\sec^{-1}(-1)$, and $\sec^{-1}(-2)$.

FUNCTION	DOMAIN	RANGE
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1} x$	$(-\infty, +\infty)$	$(0, \pi)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup [\pi, 3\pi/2)$
$\csc^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$(0, \pi/2] \cup (\pi, 3\pi/2]$

t	$\sin t$	$\cos t$	$\tan t$	$\csc t$	$\sec t$	$\cot t$
0	0	1	0	—	1	—
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	—	1	—	0

EXAMPLES: Find $\sec^{-1} 1$, $\sec^{-1}(-1)$, and $\sec^{-1}(-2)$.

Solution: We have

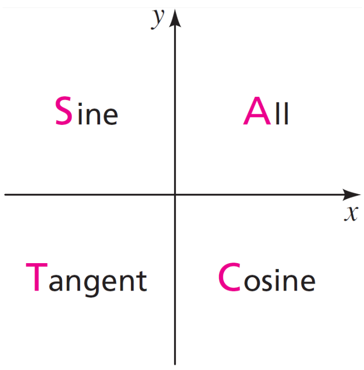
$$\sec^{-1} 1 = 0, \qquad \sec^{-1}(-1) = \pi, \qquad \sec^{-1}(-2) = \frac{4\pi}{3}$$

since

$$\sec 0 = 1, \qquad \sec \pi = -1, \qquad \sec \frac{4\pi}{3} = -2$$

and

$$0, \pi, \frac{4\pi}{3} \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$$



Note that $\sec \frac{2\pi}{3}$ is also -2, but

$$\sec^{-1}(-2) \neq \frac{2\pi}{3}$$

since

$$\frac{2\pi}{3} \notin \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$$

EXAMPLES: Find

$$\tan^{-1} 0 \qquad \cot^{-1} 0 \qquad \cot^{-1} 1 \qquad \sec^{-1} \sqrt{2} \qquad \csc^{-1} 2 \qquad \csc^{-1} \frac{2}{\sqrt{3}}$$

FUNCTION	DOMAIN	RANGE
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1} x$	$(-\infty, +\infty)$	$(0, \pi)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup [\pi, 3\pi/2)$
$\csc^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$(0, \pi/2] \cup (\pi, 3\pi/2]$

t	$\sin t$	$\cos t$	$\tan t$	$\csc t$	$\sec t$	$\cot t$
0	0	1	0	—	1	—
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	—	1	—	0

EXAMPLES: We have

$$\tan^{-1} 0 = 0, \quad \cot^{-1} 0 = \frac{\pi}{2}, \quad \cot^{-1} 1 = \frac{\pi}{4}, \quad \sec^{-1} \sqrt{2} = \frac{\pi}{4}, \quad \csc^{-1} 2 = \frac{\pi}{6}, \quad \csc^{-1} \frac{2}{\sqrt{3}} = \frac{\pi}{3}$$

EXAMPLES: Evaluate

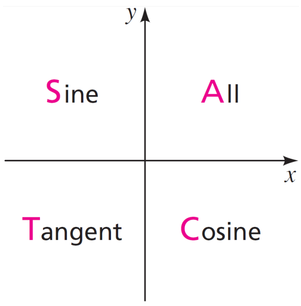
(a) $\sin\left(\arcsin \frac{\pi}{6}\right)$, $\arcsin\left(\sin \frac{\pi}{6}\right)$, and $\arcsin\left(\sin \frac{7\pi}{6}\right)$.

(b) $\sin\left(\arcsin \frac{\pi}{7}\right)$, $\arcsin\left(\sin \frac{\pi}{7}\right)$, and $\arcsin\left(\sin \frac{8\pi}{7}\right)$.

(c) $\cos\left(\arccos\left(-\frac{2}{5}\right)\right)$, $\arccos\left(\cos \frac{2\pi}{5}\right)$, and $\arccos\left(\cos \frac{9\pi}{5}\right)$.

Solution: Since $\arcsin x$ is the inverse of the restricted sine function, we have

$$\sin(\arcsin x) = x \text{ if } x \in [-1, 1] \qquad \text{and} \qquad \arcsin(\sin x) = x \text{ if } x \in [-\pi/2, \pi/2]$$



Therefore

(a) $\sin\left(\arcsin \frac{\pi}{6}\right) = \arcsin\left(\sin \frac{\pi}{6}\right) = \frac{\pi}{6}$, but

$$\arcsin\left(\sin \frac{7\pi}{6}\right) = \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

or

$$\arcsin\left(\sin \frac{7\pi}{6}\right) = \arcsin\left(\sin\left(\pi + \frac{\pi}{6}\right)\right) = \arcsin\left(-\sin \frac{\pi}{6}\right) = -\arcsin\left(\sin \frac{\pi}{6}\right) = -\frac{\pi}{6}$$

(b) $\sin\left(\arcsin \frac{\pi}{7}\right) = \arcsin\left(\sin \frac{\pi}{7}\right) = \frac{\pi}{7}$, but

$$\arcsin\left(\sin \frac{8\pi}{7}\right) = \arcsin\left(\sin\left(\frac{\pi}{7} + \pi\right)\right) = \arcsin\left(-\sin \frac{\pi}{7}\right) = -\arcsin\left(\sin \frac{\pi}{7}\right) = -\frac{\pi}{7}$$

(c) Similarly, since $\arccos x$ is the inverse of the restricted cosine function, we have

$$\cos(\arccos x) = x \text{ if } x \in [-1, 1] \qquad \text{and} \qquad \arccos(\cos x) = x \text{ if } x \in [0, \pi]$$

Therefore $\cos\left(\arccos\left(-\frac{2}{5}\right)\right) = -\frac{2}{5}$ and $\arccos\left(\cos \frac{2\pi}{5}\right) = \frac{2\pi}{5}$, but

$$\arccos\left(\cos \frac{9\pi}{5}\right) = \arccos\left(\cos\left(2\pi - \frac{\pi}{5}\right)\right) = \arccos\left(\cos \frac{\pi}{5}\right) = \frac{\pi}{5}$$