Section 5.3 Trigonometric Graphs

Graphs of the Sine and Cosine Functions

To help us graph the sine and cosine functions, we first observe that these functions repeat their values in a regular fashion. To see exactly how this happens, recall that the circumference of the unit circle is $2\pi$. It follows that the terminal point $P(x, y)$ determined by the real number $t$ is the same as that determined by $t + 2\pi$. Since the sine and cosine functions are defined in terms of the coordinates of $P(x, y)$, it follows that their values are unchanged by the addition of any integer multiple of $2\pi$. In other words,

$$\sin(t + 2n\pi) = \sin t \quad \text{for any integer } n$$
$$\cos(t + 2n\pi) = \cos t \quad \text{for any integer } n$$

Thus, the sine and cosine functions are periodic according to the following definition: A function $f$ is periodic if there is a positive number $p$ such that

$$f(t + p) = f(t) \quad \text{for every } t$$

The least such positive number (if it exists) is the period of $f$. If $f$ has period $p$, then the graph of $f$ on any interval of length $p$ is called one complete period of $f$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\sin t$</th>
<th>$\cos t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 \to $\frac{\pi}{2}$</td>
<td>0 \to 1</td>
<td>1 \to 0</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$ \to $\pi$</td>
<td>1 \to 0</td>
<td>0 \to $-1$</td>
</tr>
<tr>
<td>$\pi$ \to $\frac{3\pi}{2}$</td>
<td>0 \to $-1$</td>
<td>$-1$ \to 0</td>
</tr>
<tr>
<td>$\frac{3\pi}{2}$ \to $2\pi$</td>
<td>$-1$ \to 0</td>
<td>0 \to 1</td>
</tr>
</tbody>
</table>

So the sine and cosine functions repeat their values in any interval of length $2\pi$. To sketch their graphs, we first graph one period.

To draw the graphs more accurately, we find a few other values of $\sin t$ and $\cos t$ in the Table below. We could find still other values with the aid of a calculator.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\sin t$</th>
<th>$\cos t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
</tr>
<tr>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>1</td>
<td>$-\frac{\sqrt{3}}{2}$</td>
</tr>
<tr>
<td>$\frac{2\pi}{3}$</td>
<td>$-\frac{1}{2}$</td>
<td>$\sqrt{3}$</td>
</tr>
<tr>
<td>$\frac{5\pi}{6}$</td>
<td>0</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$-1$</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{3\pi}{2}$</td>
<td>$0$</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{5\pi}{3}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
</tr>
<tr>
<td>$\frac{11\pi}{6}$</td>
<td>$-\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Now we use this information to graph the functions \( \sin t \) and \( \cos t \) for \( t \) between 0 and \( 2\pi \). These are the graphs of one period. Using the fact that these functions are periodic with period \( 2\pi \), we get their complete graphs by continuing the same pattern to the left and to the right in every successive interval of length \( 2\pi \).

The graph of the sine function is symmetric with respect to the origin. This is as expected, since sine is an odd function. Since the cosine function is an even function, its graph is symmetric with respect to the \( y \)-axis.

**Graphs of Transformations of Sine and Cosine**

**EXAMPLE:** Sketch the graph of each function.

(a) \( f(x) = 2 + \cos x \)  
(b) \( g(x) = - \cos x \)

Solution:

(a) The graph of \( y = 2 + \cos x \) is the same as the graph of \( y = \cos x \), but shifted up 2 units.

(b) The graph of \( y = - \cos x \) is the reflection of the graph of \( y = \cos x \) in the \( x \)-axis.
Let’s graph $y = 2 \sin x$. We start with the graph of $y = \sin x$ and multiply the $y$-coordinate of each point by 2. This has the effect of stretching the graph vertically by a factor of 2. To graph $y = \frac{1}{2} \sin x$, we start with the graph of $y = \sin x$ and multiply the $y$-coordinate of each point by $\frac{1}{2}$. This has the effect of shrinking the graph vertically by a factor of $\frac{1}{2}$.

In general, for the functions

$$y = a \sin x \quad \text{and} \quad y = a \cos x$$

the number $|a|$ is called the \textbf{amplitude} and is the largest value these functions attain. Graphs of $y = a \sin x$ and $y = a \cos x$ for several values of $a$ are shown in the Figures below.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{amplitude_graphs.png}
\caption{Graphs of $y = a \sin x$ and $y = a \cos x$ for several values of $a$.}
\end{figure}

\textbf{EXAMPLE:} Find the amplitude of $y = -3 \cos x$ and sketch its graph.

\textbf{Solution:} The amplitude is $|-3| = 3$, so the largest value the graph attains is 3 and the smallest value is $-3$. To sketch the graph, we begin with the graph of $y = \cos x$, stretch the graph vertically by a factor of 3, and reflect in the $x$-axis, arriving at the graph in the Figure below.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example_graph.png}
\caption{Graph of $y = -3 \cos x$.}
\end{figure}
Since the sine and cosine functions have period $2\pi$, the functions
\[ y = a \sin kx \quad \text{and} \quad y = a \cos kx \quad (k > 0) \]
complete one period as $kx$ varies from 0 to $2\pi$, that is, for $0 \leq kx \leq 2\pi$ or for $0 \leq x \leq 2\pi/k$. So these functions complete one period as $x$ varies between 0 and $2\pi/k$ and thus have period $2\pi/k$. The graphs of these functions are called \textbf{sine curves} and \textbf{cosine curves}, respectively. (Collectively, sine and cosine curves are often referred to as \textbf{sinusoidal curves}.)

\begin{center}
\begin{tcolorbox}
\textbf{Sine and Cosine Curves}

The sine and cosine curves
\[ y = a \sin kx \quad \text{and} \quad y = a \cos kx \quad (k > 0) \]
have amplitude $|a|$ and period $2\pi/k$.

An appropriate interval on which to graph one complete period is $[0, 2\pi/k]$.
\end{tcolorbox}
\end{center}

To see how the value of $k$ affects the graph of $y = \sin kx$, let’s graph the sine curve $y = \sin 2x$. Since the period is $2\pi/2 = \pi$, the graph completes one period in the interval $0 \leq x \leq \pi$. For the sine curve $y = \sin \frac{1}{2}x$, the period is $2\pi \div \frac{1}{2} = 4\pi$, and so the graph completes one period in the interval $0 \leq x \leq 4\pi$. We see that the effect is to \textit{shrink} the graph horizontally if $k > 1$ or to \textit{stretch} the graph horizontally if $k < 1$.

For comparison, in the Figure below we show the graphs of one period of the sine curve $y = a \sin kx$ for several values of $k$.

\begin{center}
\end{center}

\begin{center}
\textbf{EXAMPLE:} Find the amplitude and period of each function, and sketch its graph.

(a) $y = 4 \cos 3x$ \quad (b) $y = -2 \sin \frac{1}{2}x$
\end{center}
EXAMPLE: Find the amplitude and period of each function, and sketch its graph.

(a) \( y = 4 \cos 3x \)  
(b) \( y = -2 \sin \frac{1}{2}x \)

Solution:
(a) We get the amplitude and period from the form of the function as follows:

\[
\text{amplitude} = |a| = 4 \\
\text{period} = \frac{2\pi}{k} = \frac{2\pi}{3}
\]

The amplitude is 4 and the period is \(2\pi/3\). The graph is shown in the Figure below.

(b) For \( y = -2 \sin \frac{1}{2}x \),

\[
\text{amplitude} = |a| = |-2| = 2 \\
\text{period} = \frac{2\pi}{k} = \frac{2\pi}{1/2} = 4\pi
\]

The graph is shown in the Figure above.

EXAMPLE: Find the amplitude and period of each function, and sketch its graph.

(a) \( y = 2 \sin \pi x \)  
(b) \( y = 3 \sin 2x + 1 \)  
(c) \( y = 3 \cos 2x + 2 \)
EXAMPLE: Find the amplitude and period of each function, and sketch its graph.

(a) \( y = 2 \sin \pi x \)  
(b) \( y = 3 \sin 2x + 1 \)  
(c) \( y = 3 \cos 2x + 2 \)

Solution: We have

(a) amplitude = \(|2| = 2\)  
(b) amplitude = \(|3| = 3\)  
(c) amplitude = \(|3| = 3\)

\[
\text{period } = \frac{2\pi}{k} = \frac{2\pi}{\pi} = 2
\]

\[
\text{period } = \frac{2\pi}{k} = \frac{2\pi}{2} = \pi
\]

\[
\text{period } = \frac{2\pi}{k} = \frac{2\pi}{2} = \pi
\]

The graphs of functions of the form

\[ y = a \sin k(x - b) \quad \text{and} \quad y = a \cos k(x - b) \quad (k > 0) \]

are simply sine and cosine curves shifted horizontally by an amount \(|b|\). They are shifted to the right if \(b > 0\) or to the left if \(b < 0\). The number \(b\) is the \textit{phase shift}. We summarize the properties of these functions in the following box.

**Shifted Sine and Cosine Curves**

The sine and cosine curves

\[ y = a \sin k(x - b) \quad \text{and} \quad y = a \cos k(x - b) \quad (k > 0) \]

have amplitude \(|a|\), period \(2\pi/k\), and phase shift \(b\).

An appropriate interval on which to graph one complete period is \([b, b + (2\pi/k)]\).

EXAMPLE: The graphs of \(y = \sin \left(x - \frac{\pi}{3}\right)\), \(y = \sin \left(x + \frac{\pi}{6}\right)\), and \(y = \frac{1}{2} \sin \left(x - \frac{\pi}{3}\right)\) are shown in the Figures below.

EXAMPLE: Find the amplitude, the period, and the phase shift of

(a) \( y = 2 \cos 3x \)  
(b) \( y = -5 \cos \frac{1}{3}x \)  
(c) \( y = -2 \sin 4(x - 1.3) \)
EXAMPLE: Find the amplitude, the period, and the phase shift of

(a) \( y = 2 \cos 3x \)  
(b) \( y = -5 \cos \frac{1}{3}x \)  
(c) \( y = -2 \sin 4(x - 1.3) \)

Solution: We have

(a) amplitude = \(|2| = 2\)  
period = \(\frac{2\pi}{3}\)  
phase shift = \(b = 0\)

(b) amplitude = \(|-5| = 5\)  
period = \(\frac{2\pi}{1/3} = 6\pi\)  
phase shift = \(b = 0\)

(c) amplitude = \(|-2| = 2\)  
period = \(\frac{2\pi}{4} = \frac{\pi}{2}\)  
phase shift = \(b = 1.3\)  
(Shift 1.3 to the right)

EXAMPLE: Find the amplitude, period, and phase shift of

\( y = 3 \sin 2 \left(x - \frac{\pi}{4}\right) \)

and graph one complete period.

Solution 1: We get the amplitude, period, and phase shift from the form of the function as follows:

\[
\begin{align*}
\text{amplitude} &= |a| = 3 \\
\text{period} &= \frac{2\pi}{k} = \frac{2\pi}{2} = \pi \\
y &= 3 \sin 2 \left(x - \frac{\pi}{4}\right) \\
\text{phase shift} &= \frac{\pi}{4} \text{ (to the right)}
\end{align*}
\]

Since the phase shift is \(\pi /4\) and the period is \(\pi\), one complete period occurs on the interval

\[
\left[\frac{\pi}{4}, \frac{3\pi}{4} + \pi\right] = \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]
\]

As an aid in sketching the graph, we divide this interval into four equal parts, then graph a sine curve with amplitude 3 as in the Figure below.

![Graph of y = 3 sin 2(x - \pi/4)]

Solution 2: Since the period of \(y = \sin x\) is \(2\pi\), the function \(y = 3 \sin 2 \left(x - \frac{\pi}{4}\right)\) will go through one complete period as \(2 \left(x - \frac{\pi}{4}\right)\) varies from 0 to \(2\pi\).

Start of period:  \(2 \left(x - \frac{\pi}{4}\right) = 0\)  
\(x - \frac{\pi}{4} = 0\)  
\(x = \frac{\pi}{4}\)

End of period:  \(2 \left(x - \frac{\pi}{4}\right) = 2\pi\)  
\(x - \frac{\pi}{4} = \pi\)  
\(x = \frac{5\pi}{4}\)

So we graph one period on the interval \(\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]\).
EXAMPLE: Find the amplitude, period, and phase shift of

\[ y = \frac{1}{2} \cos \left( \frac{\pi}{3} (x - 1) \right) \]

and graph one complete period.

Solution: We have

amplitude = \( |a| = \frac{1}{2} \)

period = \( \frac{2\pi}{k} = \frac{2\pi}{\pi/3} = 6 \)

phase shift = \( b = 1 \) (Shift 1 to the right)

From this information it follows that one period of this cosine curve begins at 1 and ends at \( 1 + 6 = 7 \).

EXAMPLE: Find the amplitude, period, and phase shift of

\[ y = \frac{3}{4} \cos \left( 2x + \frac{2\pi}{3} \right) \]

and graph one complete period.
EXAMPLE: Find the amplitude, period, and phase shift of
\[ y = \frac{3}{4} \cos \left( 2x + \frac{2\pi}{3} \right) \]
and graph one complete period.

Solution: We first write this function in the form \( y = a \cos k(x - b) \). To do this, we factor 2 from the expression \( 2x + \frac{2\pi}{3} \) to get
\[ y = \frac{3}{4} \cos 2 \left[ x - \left( -\frac{\pi}{3} \right) \right] \]
Thus we have
\[
\text{amplitude} = |a| = \frac{3}{4} \\
\text{period} = \frac{2\pi}{k} = \frac{2\pi}{2} = \pi \\
\text{phase shift} = b = -\frac{\pi}{3} \quad \text{(Shift } \frac{\pi}{3} \text{ to the left)}
\]
From this information it follows that one period of this cosine curve begins at \(-\pi/3\) and ends at \((-\pi/3) + \pi = 2\pi/3\). To sketch the graph over the interval \([-\pi/3, 2\pi/3]\), we divide this interval into four equal parts and graph a cosine curve with amplitude \(\frac{3}{4}\) as shown in the Figure below.

We can also find one complete period as follows:

<table>
<thead>
<tr>
<th>Start of period:</th>
<th>End of period:</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2x + \frac{2\pi}{3} = 0)</td>
<td>(2x + \frac{2\pi}{3} = 2\pi)</td>
</tr>
<tr>
<td>(2x = -\frac{2\pi}{3})</td>
<td>(2x = \frac{4\pi}{3})</td>
</tr>
<tr>
<td>(x = -\frac{\pi}{3})</td>
<td>(x = \frac{2\pi}{3})</td>
</tr>
</tbody>
</table>

So we graph one period on the interval \([-\pi/3, 2\pi/3]\).

EXAMPLE: Find the amplitude, period, phase shift and graph one complete period of each function.

(a) \( y = \frac{\sqrt{2}}{2} \cos (3x + 2) \)  
(b) \( y = -3 \cos (2\pi x + 4\pi) \)
EXAMPLE: Find the amplitude, period, phase shift and graph one complete period of each function.

(a) \( y = \frac{\sqrt{2}}{2} \cos (3x + 2) \)  
(b) \( y = -3 \cos (2\pi x + 4\pi) \)

Solution:

(a) We first write this function in the form \( y = a \cos k(x - b) \). To do this, we factor 3 from the expression \( 3x + 2 \) to get

\[
y = \frac{\sqrt{2}}{2} \cos 3 \left[ x - \left( -\frac{2}{3} \right) \right]
\]

Thus, we have

amplitude = \( |a| = \frac{\sqrt{2}}{2} \)

period = \( \frac{2\pi}{k} = \frac{2\pi}{3} \)

phase shift = \( b = -\frac{2}{3} \) (Shift \( \frac{2}{3} \) to the left)

From this information it follows that one period of this cosine curve begins at \(-2/3\) and ends at \(-2/3 + 2\pi/3 = 2(\pi - 1)/3\).

(b) We first write this function in the form \( y = a \cos k(x - b) \). To do this, we factor 3 from the expression \( 2\pi x + 4\pi \) to get

\[
y = -3 \cos 2\pi \left( x - (-\frac{2}{3}) \right)
\]

Thus, we have

amplitude = \( |a| = |-3| = 3 \)

period = \( \frac{2\pi}{k} = \frac{2\pi}{2\pi} = 1 \)

phase shift = \( b = -2 \) (Shift 2 to the left)

From this information it follows that one period of this cosine curve begins at \(-2\) and ends at \(-2 + 1 = -1\).

EXAMPLE: Find the amplitude, period, phase shift and graph one complete period of the function

\[
y = 2 \cos \left( \frac{1}{2} x + 1 \right)
\]
EXAMPLE: Find the amplitude, period, phase shift and graph one complete period of the function

\[ y = 2 \cos \left( \frac{1}{2} x + 1 \right) \]

Solution: We first write this function in the form \( y = a \cos k(x - b) \). To do this, we factor \( \frac{1}{2} \) from the expression \( \frac{1}{2} x + 1 \) to get

\[ y = 2 \cos \frac{1}{2} [x - (-2)] \]

Thus we have

amplitude = \(|a| = 2\)

period = \( \frac{2\pi}{k} = \frac{2\pi}{1/2} = 4\pi \)

phase shift = \( b = -2 \) (Shift 2 to the left)

From this information it follows that one period of this cosine curve begins at \(-2\) and ends at \(-2 + 4\pi\). To sketch the graph over the interval \([-2, -2 + 4\pi]\), we divide this interval into four equal parts and graph a cosine curve with amplitude 2 as shown in the Figure below.

We can also find one complete period as follows:

Start of period: \( \frac{1}{2} x + 1 = 0 \) \( \frac{1}{2} x + 1 = 2\pi \)

\( \frac{1}{2} x = -1 \) \( \frac{1}{2} x = -1 + 2\pi \)

\( x = -2 \) \( x = -2 + 4\pi \)

So we graph one period on the interval \([-2, -2 + 4\pi]\).

We finally note that the \( x \)-intercepts are \( x = \pi - 2 \) and \( x = 3\pi - 2 \).

EXAMPLE: Find the amplitude, period, phase shift and graph one complete period of the function

\[ y = \frac{1}{2} \sin (2x - 1) \]
EXAMPLE: Find the amplitude, period, phase shift and graph one complete period period of the function

\[ y = \frac{1}{2} \sin (2x - 1) \]

Solution: We first write this function in the form \( y = a \sin k(x - b) \). To do this, we factor 2 from the expression \( 2x - 1 \) to get

\[ y = \frac{1}{2} \sin 2 \left( x - \frac{1}{2} \right) \]

Thus we have

- **amplitude** = \( |a| = \frac{1}{2} \)
- **period** = \( \frac{2\pi}{k} = \frac{2\pi}{2} = \pi \)
- **phase shift** = \( b = \frac{1}{2} \) (Shift \( \frac{1}{2} \) to the right)

From this information it follows that one period of this sine curve begins at \( \frac{1}{2} \) and ends at \( \frac{1}{2} + \pi \). To sketch the graph over the interval \( \left[ \frac{1}{2}, \frac{1}{2} + \pi \right] \), we divide this interval into four equal parts and graph a sine curve with amplitude \( \frac{1}{2} \) as shown in the Figure below.

We can also find one complete period as follows:

Start of period: \( 2x - 1 = 0 \) \quad End of period: \( 2x - 1 = 2\pi \)

\[ 2x = 1 \quad 2x = 1 + 2\pi \]

\[ x = \frac{1}{2} \quad x = \frac{1}{2} + \pi \]

So we graph one period on the interval \( \left[ \frac{1}{2}, \frac{1}{2} + \pi \right] \).

We finally note that the \( x \)-intercepts are \( x = \frac{1}{2}, \frac{1}{2} + \frac{\pi}{2}, \) and \( \frac{1}{2} + \pi \).