Section 5.2 Trigonometric Functions of Real Numbers

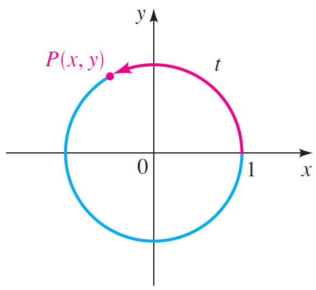
The Trigonometric Functions

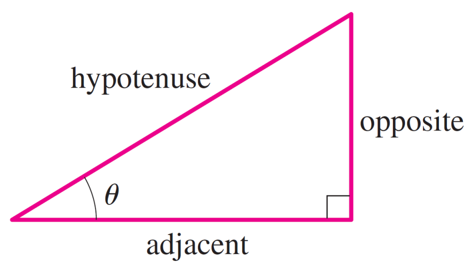
Definition of the Trigonometric Functions

Let t be any real number and let $P(x, y)$ be the terminal point on the unit circle determined by t. We define

$$\sin t = y \qquad \cos t = x \qquad \tan t = \frac{y}{x} \quad (x \neq 0)$$

$$\csc t = \frac{1}{y} \quad (y \neq 0) \qquad \sec t = \frac{1}{x} \quad (x \neq 0) \qquad \cot t = \frac{x}{y} \quad (y \neq 0)$$





The Trigonometric Ratios

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

EXAMPLE: Use the Table below to find the six trigonometric functions of each given real number t.

(a) $t = \frac{\pi}{3}$ (b) $t = \frac{\pi}{2}$

t	Terminal point determined by t
0	$(1,0)$
$\frac{\pi}{6}$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{3}$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{2}$	$(0,1)$

EXAMPLE: Use the Table below to find the six trigonometric functions of each given real number t.

(a) $t = \dfrac{\pi}{3}$ (b) $t = \dfrac{\pi}{2}$

t	Terminal point determined by t
0	$(1,0)$
$\frac{\pi}{6}$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{3}$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{2}$	$(0,1)$

Solution:

(a) From the Table, we see that the terminal point determined by $t = \pi/3$ is $P(1/2, \sqrt{3}/2)$. Since the coordinates are $x = 1/2$ and $y = \sqrt{3}/2$, we have

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2} \qquad \tan\frac{\pi}{3} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

$$\csc\frac{\pi}{3} = \frac{2\sqrt{3}}{3} \qquad \sec\frac{\pi}{3} = 2 \qquad \cot\frac{\pi}{3} = \frac{1/2}{\sqrt{3}/2} = \frac{\sqrt{3}}{3}$$

(b) The terminal point determined by $\pi/2$ is $P(0,1)$. So

$$\sin\frac{\pi}{2} = 1 \qquad \cos\frac{\pi}{2} = 0 \qquad \csc\frac{\pi}{2} = \frac{1}{1} = 1 \qquad \cot\frac{\pi}{2} = \frac{0}{1} = 0$$

But $\tan\pi/2$ and $\sec\pi/2$ are undefined because $x = 0$ appears in the denominator in each of their definitions.

EXAMPLE: Find the six trigonometric functions of each given real number $t = \dfrac{\pi}{4}$.

Solution: From the Table above, we see that the terminal point determined by $t = \pi/4$ is $P(\sqrt{2}/2, \sqrt{2}/2)$. Since the coordinates are $x = \sqrt{2}/2$ and $y = \sqrt{2}/2$, we have

$$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \tan\frac{\pi}{4} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$$

$$\csc\frac{\pi}{4} = \sqrt{2} \qquad \sec\frac{\pi}{4} = \sqrt{2} \qquad \cot\frac{\pi}{4} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$$

Table 1 Special values of the trigonometric functions

t	$\sin t$	$\cos t$	$\tan t$	$\csc t$	$\sec t$	$\cot t$
0	0	1	0	—	1	—
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	—	1	—	0

We can easily remember the sines and cosines of the basic angles by writing them in the form $\sqrt{\square}/2$:

t	$\sin t$	$\cos t$
0	$\sqrt{0}/2$	$\sqrt{4}/2$
$\pi/6$	$\sqrt{1}/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	$\sqrt{1}/2$
$\pi/2$	$\sqrt{4}/2$	$\sqrt{0}/2$

Domains of the Trigonometric Functions

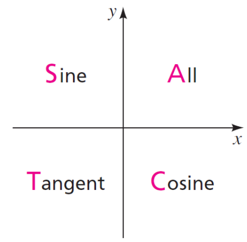
Function	Domain
sin, cos	All real numbers
tan, sec	All real numbers other than $\frac{\pi}{2} + n\pi$ for any integer n
cot, csc	All real numbers other than $n\pi$ for any integer, n

Values of the Trigonometric Functions

Signs of the Trigonometric Functions

Quadrant	Positive Functions	Negative functions
I	all	none
II	sin, csc	cos, sec, tan, cot
III	tan, cot	sin, csc, cos, sec
IV	cos, sec	sin, csc, tan, cot

The following mnemonic device will help you remember which trigonometric functions are positive in each quadrant: **A**ll of them, **S**ine, **T**angent, or **C**osine.



You can remember this as "All Students Take Calculus."

EXAMPLE:

(a) $\cos \frac{\pi}{3} > 0$, because the terminal point of $t = \frac{\pi}{3}$ is in Quadrant I.

(b) $\tan 4 > 0$, because the terminal point of $t = 4$ is in Quadrant III.

(c) If $\cos t < 0$ and $\sin t > 0$, then the terminal point of t must be in Quadrant II.

EXAMPLE: Determine the sign of each function.

(a) $\cos \frac{7\pi}{4}$ (b) $\tan 1$

Solution:

(a) Positive (b) Positive

EXAMPLE: Find each value.

(a) $\cos \frac{2\pi}{3}$ (b) $\tan \left(-\frac{\pi}{3}\right)$ (c) $\sin \frac{19\pi}{4}$

EXAMPLE: Find each value.

(a) $\cos \dfrac{2\pi}{3}$ (b) $\tan\left(-\dfrac{\pi}{3}\right)$ (c) $\sin \dfrac{19\pi}{4}$

Solution:

(a) Since

$$\frac{2\pi}{3} = \frac{3\pi-\pi}{3} = \frac{3\pi}{3} - \frac{\pi}{3} = \pi - \frac{\pi}{3}$$

the reference number for $2\pi/3$ is $\pi/3$ (see Figure (a) below) and the terminal point of $2\pi/3$ is in Quadrant II. Thus $\cos(2\pi/3)$ is negative and

$$\cos\frac{2\pi}{3} = -\cos\frac{\pi}{3} = -\frac{1}{2}$$

Sign Reference number From Table 1

(b) The reference number for $-\pi/3$ is $\pi/3$ (see Figure (b) below). Since the terminal point of $-\pi/3$ is in Quadrant IV, $\tan(-\pi/3)$ is negative. Thus

$$\tan\left(-\frac{\pi}{3}\right) = -\tan\frac{\pi}{3} = -\sqrt{3}$$

Sign Reference number From Table 1

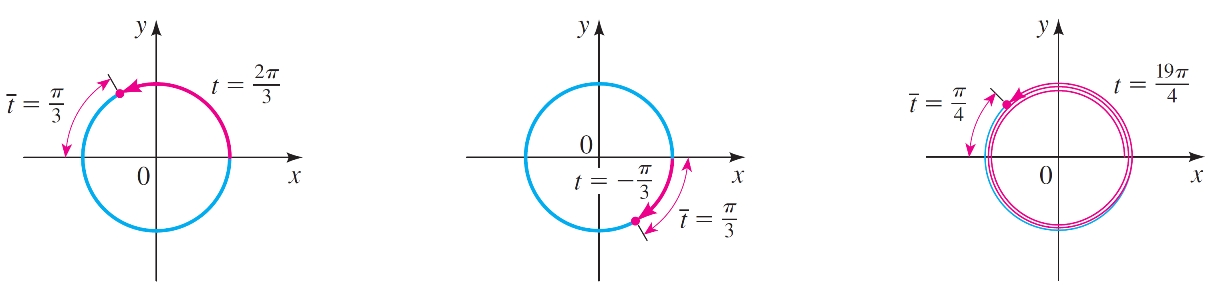
(c) Since

$$\frac{19\pi}{4} = \frac{20\pi-\pi}{4} = \frac{20\pi}{4} - \frac{\pi}{4} = 5\pi - \frac{\pi}{4}$$

the reference number for $19\pi/4$ is $\pi/4$ (see Figure (c) below) and the terminal point of $19\pi/4$ is in Quadrant II. Thus $\sin(19\pi/4)$ is positive and

$$\sin\frac{19\pi}{4} = +\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

Sign Reference number From Table 1



EXAMPLE: Find each value.

(a) $\sin \dfrac{2\pi}{3}$ (b) $\tan\left(-\dfrac{4\pi}{3}\right)$ (c) $\cos \dfrac{14\pi}{3}$

EXAMPLE: Find each value.

(a) $\sin\dfrac{2\pi}{3}$ (b) $\tan\left(-\dfrac{4\pi}{3}\right)$ (c) $\cos\dfrac{14\pi}{3}$

Solution:

(a) Since

$$\frac{2\pi}{3}=\frac{3\pi-\pi}{3}=\frac{3\pi}{3}-\frac{\pi}{3}=\pi-\frac{\pi}{3}$$

the reference number for $2\pi/3$ is $\pi/3$ and the terminal point of $2\pi/3$ is in Quadrant II. Thus $\sin(2\pi/3)$ is positive and

$$\sin\frac{2\pi}{3}=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$$

(b) Since

$$-\frac{4\pi}{3}=-\frac{3\pi+\pi}{3}=-\frac{3\pi}{3}-\frac{\pi}{3}=-\pi-\frac{\pi}{3}$$

the reference number for $-4\pi/3$ is $\pi/3$ and the terminal point of $-4\pi/3$ is in Quadrant II. Thus $\tan(-4\pi/3)$ is negative and

$$\tan\left(-\frac{4\pi}{3}\right)=-\tan\left(\frac{\pi}{3}\right)=-\sqrt{3}$$

(c) Since

$$\frac{14\pi}{3}=\frac{15\pi-\pi}{3}=\frac{15\pi}{3}-\frac{\pi}{3}=5\pi-\frac{\pi}{3}$$

the reference number for $14\pi/3$ is $\pi/3$ and the terminal point of $14\pi/3$ is in Quadrant II. Thus $\cos(14\pi/4)$ is negative and

$$\cos\frac{14\pi}{3}=-\cos\frac{\pi}{3}=-\frac{1}{2}$$

EXAMPLE: Evaluate

(a) $\sin\dfrac{\pi}{3}$ (b) $\cos\dfrac{7\pi}{6}$ (c) $\tan\dfrac{11\pi}{4}$ (d) $\sec\dfrac{17\pi}{3}$ (e) $\csc\dfrac{17\pi}{2}$ (f) $\cot\dfrac{121\pi}{6}$

EXAMPLE: Evaluate

(a) $\sin\dfrac{\pi}{3}$ (b) $\cos\dfrac{7\pi}{6}$ (c) $\tan\dfrac{11\pi}{4}$ (d) $\sec\dfrac{17\pi}{3}$ (e) $\csc\dfrac{17\pi}{2}$ (f) $\cot\dfrac{121\pi}{6}$

Solution:

(a) The reference number for $\pi/3$ is $\pi/3$. Since the terminal point of $\pi/3$ is in Quadrant I, $\sin(\pi/3)$ is positive. Thus

$$\sin\frac{\pi}{3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

(b) Since $\frac{7\pi}{6} = \frac{6\pi+\pi}{6} = \frac{6\pi}{6} + \frac{\pi}{6} = \pi + \frac{\pi}{6}$, the reference number for $7\pi/6$ is $\pi/6$ and the terminal point of $7\pi/6$ is in Quadrant III. Thus $\cos(7\pi/6)$ is negative and

$$\cos\frac{7\pi}{6} = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$

(c) Since $\frac{11\pi}{4} = \frac{12\pi-\pi}{4} = \frac{12\pi}{4} - \frac{\pi}{4} = 3\pi - \frac{\pi}{4}$, the reference number for $11\pi/4$ is $\pi/4$ and the terminal point of $11\pi/6$ is in Quadrant II. Thus $\tan(11\pi/4)$ is negative and

$$\tan\frac{11\pi}{4} = -\tan\frac{\pi}{4} = -1$$

(d) Since $\frac{17\pi}{3} = \frac{18\pi-\pi}{3} = \frac{18\pi}{3} - \frac{\pi}{3} = 6\pi - \frac{\pi}{3}$, the reference number for $17\pi/3$ is $\pi/3$ and the terminal point of $17\pi/3$ is in Quadrant IV. Thus $\sec(17\pi/3)$ is positive and

$$\sec\frac{17\pi}{3} = \sec\frac{\pi}{3} = 2$$

(e) Since $\frac{17\pi}{2} = \frac{16\pi+\pi}{2} = \frac{16\pi}{2} + \frac{\pi}{2} = 8\pi + \frac{\pi}{2}$, the reference number for $17\pi/2$ is $\pi/2$ and the terminal point of $17\pi/2$ is in Quadrant I (II). Thus $\csc(17\pi/2)$ is positive and

$$\csc\frac{17\pi}{2} = \csc\frac{\pi}{2} = 1$$

(f) Since $\frac{121\pi}{6} = \frac{120\pi+\pi}{6} = \frac{120\pi}{6} + \frac{\pi}{6} = 20\pi + \frac{\pi}{6}$, the reference number for $121\pi/6$ is $\pi/6$ and the terminal point of $121\pi/6$ is in Quadrant I. Thus $\cot(121\pi/6)$ is positive and

$$\cot\frac{121\pi}{6} = \cot\frac{\pi}{6} = \sqrt{3}$$

Even-Odd Properties

Sine, cosecant, tangent, and cotangent are odd functions; cosine and secant are even functions.

$$\sin(-t) = -\sin t \qquad \cos(-t) = \cos t \qquad \tan(-t) = -\tan t$$

$$\csc(-t) = -\csc t \qquad \sec(-t) = \sec t \qquad \cot(-t) = -\cot t$$

EXAMPLE: Use the even-odd properties of the trigonometric functions to determine each value.

(a) $\sin\left(-\dfrac{\pi}{6}\right)$ (b) $\cos\left(-\dfrac{\pi}{4}\right)$ (c) $\csc\left(-\dfrac{\pi}{3}\right)$

EXAMPLE: Use the even-odd properties of the trigonometric functions to determine each value.

(a) $\sin\left(-\frac{\pi}{6}\right)$ (b) $\cos\left(-\frac{\pi}{4}\right)$ (c) $\csc\left(-\frac{\pi}{3}\right)$

Solution: We have

(a) $\sin\left(-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2}$ (b) $\cos\left(-\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

(c) $\csc\left(-\frac{\pi}{3}\right) = -\csc\frac{\pi}{3} = -\frac{2\sqrt{3}}{3}$

Fundamental Identities

Fundamental Identities

Reciprocal Identities

$$\csc t = \frac{1}{\sin t} \qquad \sec t = \frac{1}{\cos t} \qquad \cot t = \frac{1}{\tan t}$$

$$\tan t = \frac{\sin t}{\cos t} \qquad \cot t = \frac{\cos t}{\sin t}$$

Pythagorean Identities

$$\sin^2 t + \cos^2 t = 1 \qquad \tan^2 t + 1 = \sec^2 t \qquad 1 + \cot^2 t = \csc^2 t$$

Proof: The reciprocal identities follow immediately from the definition. We now prove the Pythagorean identities. By definition, $\cos t = x$ and $\sin t = y$, where x and y are the coordinates of a point $P(x, y)$ on the unit circle. Since $P(x, y)$ is on the unit circle, we have $x^2 + y^2 = 1$. Thus

$$\sin^2 t + \cos^2 t = 1$$

Dividing both sides by $\cos^2 t$ (provided $\cos t \neq 0$), we get

$$\frac{\sin^2 t}{\cos^2 t} + \frac{\cos^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$$

$$\left(\frac{\sin^2 t}{\cos^2 t}\right)^2 + 1 = \left(\frac{1}{\cos^2 t}\right)^2$$

$$\tan^2 t + 1 = \sec^2 t$$

We have used the reciprocal identities $\sin t/\cos t = \tan t$ and $1/\cos t = \sec t$. Similarly, dividing both sides of the first Pythagorean identity by $\sin^2 t$ (provided $\sin t \neq 0$) gives us $1 + \cot^2 t = \csc^2 t$.

EXAMPLE: If $\cos t = \frac{3}{5}$ and t is in Quadrant IV, find the values of all the trigonometric functions at t.

EXAMPLE: If $\cos t = \frac{3}{5}$ and t is in Quadrant IV, find the values of all the trigonometric functions at t.

Solution: From the Pythagorean identities we have

$$\sin^2 t + \cos^2 t = 1$$

$$\sin^2 t + \left(\frac{3}{5}\right)^2 = 1$$

$$\sin^2 t = 1 - \frac{9}{25} = \frac{16}{25}$$

$$\sin t = \pm\frac{4}{5}$$

Since this point is in Quadrant IV, $\sin t$ is negative, so $\sin t = -\frac{4}{5}$. Now that we know both $\sin t$ and $\cos t$, we can find the values of the other trigonometric functions using the reciprocal identities:

$$\sin t = -\frac{4}{5} \qquad \cos t = \frac{3}{5} \qquad \tan t = \frac{\sin t}{\cos t} = \frac{-\frac{4}{5}}{\frac{3}{5}} = -\frac{4}{3}$$

$$\csc t = \frac{1}{\sin t} = -\frac{5}{4} \qquad \sec t = \frac{1}{\cos t} = \frac{5}{3} \qquad \cot t = \frac{1}{\tan t} = -\frac{3}{4}$$

EXAMPLE: If $\cos t = -\frac{5}{13}$ and t is in Quadrant II, find the values of all the trigonometric functions at t.

Solution: From the Pythagorean identities we have

$$\sin^2 t + \cos^2 t = 1$$

$$\sin^2 t + \left(-\frac{5}{13}\right)^2 = 1$$

$$\sin^2 t = 1 - \frac{25}{169} = \frac{144}{169}$$

$$\sin t = \pm\frac{12}{13}$$

Since this point is in Quadrant II, $\sin t$ is positive, so $\sin t = \frac{12}{13}$. Now that we know both $\sin t$ and $\cos t$, we can find the values of the other trigonometric functions using the reciprocal identities:

$$\sin t = \frac{12}{13} \qquad \cos t = -\frac{5}{13} \qquad \tan t = \frac{\sin t}{\cos t} = \frac{\frac{12}{13}}{-\frac{5}{13}} = -\frac{12}{5}$$

$$\csc t = \frac{1}{\sin t} = \frac{13}{12} \qquad \sec t = \frac{1}{\cos t} = -\frac{13}{5} \qquad \cot t = \frac{1}{\tan t} = -\frac{5}{12}$$

EXAMPLE: Write $\tan t$ in terms of $\cos t$, where t is in Quadrant III.

EXAMPLE: Write $\tan t$ in terms of $\cos t$, where t is in Quadrant III.

Solution: Since $\tan t = \sin t / \cos t$, we need to write $\sin t$ in terms of $\cos t$. By the Pythagorean identities we have

$$\sin^2 t + \cos^2 t = 1$$

$$\sin^2 t = 1 - \cos^2 t$$

$$\sin t = \pm\sqrt{1 - \cos^2 t}$$

Since $\sin t$ is negative in Quadrant III, the negative sign applies here. Thus

$$\tan t = \frac{\sin t}{\cos t} = -\frac{\sqrt{1-\cos^2 t}}{\cos t}$$

EXAMPLE: Write $\tan t$ in terms of $\cos t$, where t is in Quadrant I.

Solution: Since $\tan t = \sin t / \cos t$, we need to write $\sin t$ in terms of $\cos t$. By the Pythagorean identities we have

$$\sin^2 t + \cos^2 t = 1$$

$$\sin^2 t = 1 - \cos^2 t$$

$$\sin t = \pm\sqrt{1 - \cos^2 t}$$

Since $\sin t$ is positive in Quadrant I, the positive sign applies here. Thus

$$\tan t = \frac{\sin t}{\cos t} = \frac{\sqrt{1-\cos^2 t}}{\cos t}$$

EXAMPLE: Write $\cos t$ in terms of $\tan t$, where t is in Quadrant II.

Solution: Since $\tan t = \sin t / \cos t$, we need to write $\sin t$ in terms of $\cos t$. By the Pythagorean identities we have

$$\sin^2 t + \cos^2 t = 1$$

$$\sin^2 t = 1 - \cos^2 t$$

so $\tan^2 t = \dfrac{\sin^2 t}{\cos^2 t} = \dfrac{1-\cos^2 t}{\cos^2 t}$. Multiplying both sides by $\cos^2 t$, we get

$$\cos^2 t \tan^2 t = 1 - \cos^2 t$$

$$\cos^2 t \tan^2 t + \cos^2 t = 1$$

$$\cos^2 t(\tan^2 t + 1) = 1$$

$$\cos^2 t = \frac{1}{\tan^2 t + 1}$$

$$\cos t = \pm\frac{1}{\sqrt{\tan^2 t + 1}}$$

Since $\cos t$ is negative in Quadrant II, the negative sign applies here. Thus

$$\cos t = -\frac{1}{\sqrt{\tan^2 t + 1}}$$