

Section 4.6 Modeling with Exponential and Logarithmic Functions

Many processes that occur in nature, such as population growth, radioactive decay, heat diffusion, and numerous others, can be modeled using exponential functions. Logarithmic functions are used in models for the loudness of sounds, the intensity of earthquakes, and many other phenomena.

Exponential Models of Population Growth

Biologists have observed that the population of a species doubles its size in a fixed period of time. For example, under ideal conditions a certain population of bacteria doubles in size every 3 hours. If the culture is started with 1000 bacteria, then after 3 hours there will be 2000 bacteria, after another 3 hours there will be 4000, and so on. If we let $n = n(t)$ be the number of bacteria after t hours, then

$$\begin{aligned}n(0) &= 1000 \\n(3) &= 1000 \cdot 2 \\n(6) &= (1000 \cdot 2) \cdot 2 = 1000 \cdot 2^2 \\n(9) &= (1000 \cdot 2^2) \cdot 2 = 1000 \cdot 2^3 \\n(12) &= (1000 \cdot 2^3) \cdot 2 = 1000 \cdot 2^4\end{aligned}$$

From this pattern it appears that the number of bacteria after t hours is modeled by the function

$$n(t) = 1000 \cdot 2^{t/3}$$

In general, suppose that the initial size of a population is n_0 and the doubling period is a . Then the size of the population at time t is modeled by

$$n(t) = n_0 2^{ct}$$

where $c = 1/a$. If we knew the tripling time b , then the formula would be $n(t) = n_0 3^{ct}$ where $c = 1/b$. These formulas indicate that the growth of the bacteria is modeled by an exponential function. But what base should we use? The answer is e , because then it can be shown (using calculus) that the population is modeled by

$$n(t) = n_0 e^{rt}$$

where r is the *relative rate of growth of population, expressed as a proportion of the population at any time*. For instance, if $r = 0.02$, then at any time t the growth rate is 2% of the population at time t .

REMARK: Notice that the formula for population growth is the same as that for continuously compounded interest. In fact, the same principle is at work in both cases: The growth of a population (or an investment) per time period is proportional to the size of the population (or the amount of the investment). A population of 1,000,000 will increase more in one year than a population of 1000; in exactly the same way, an investment of \$1,000,000 will increase more in one year than an investment of \$1000.

EXPONENTIAL GROWTH MODEL: A population that experiences **exponential growth** increases according to the model

$$n(t) = n_0 e^{rt}$$

where t is time, $n(t)$ is the population size at time t , n_0 is the initial size of the population, and r is the relative rate of growth (expressed as a proportion of the population).

EXAMPLE: The initial bacterium count in a culture is 500. A biologist later makes a sample count of bacteria in the culture and finds that the relative rate of growth is 40% per hour.

- (a) Find a function that models the number of bacteria after t hours.
- (b) What is the estimated count after 1 hour?
- (c) What is the estimated count after 10 hours?
- (d) Sketch the graph of the function $n(t)$.

Solution:

- (a) We use the exponential growth model with $n_0 = 500$ and $r = 0.4$ to get

$$n(t) = 500e^{0.4t}$$

where t is measured in hours.

- (b) Using the function in part (a), we find that the bacterium count after 1 hour is

$$n(1) = 500e^{0.4(1)} = 500e^{0.4} \approx 746$$

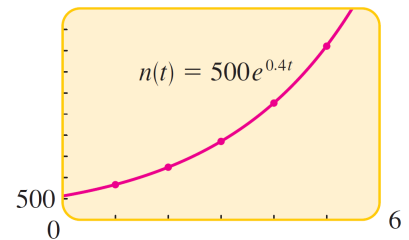
Note that $n(1) \neq 500(1 + 0.4) = 700$.

- (c) Using the function in part (a), we find that the bacterium count after 10 hours is

$$n(10) = 500e^{0.4(10)} = 500e^4 \approx 27,300$$

Note that $n(10) \neq 500(1 + 0.4)^{10} \approx 14,463$.

- (d) The graph is shown.



EXAMPLE: In 2000 the population of the world was 6.1 billion and the relative rate of growth was 1.4% per year. It is claimed that a rate of 1.0% per year would make a significant difference in the total population in just a few decades. Test this claim by estimating the population of the world in the year 2050 using a relative rate of growth of (a) 1.4% per year and (b) 1.0% per year. Graph the population functions for the next 100 years for the two relative growth rates in the same viewing rectangle.

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Solution:

(a) By the exponential growth model, we have

$$n(t) = 6.1e^{0.014t}$$

where $n(t)$ is measured in billions and t is measured in years since 2000. Because the year 2050 is 50 years after 2000, we find

$$n(50) = 6.1e^{0.014(50)} = 6.1e^{0.7} \approx 12.3$$

The estimated population in the year 2050 is about 12.3 billion.

(b) We use the function

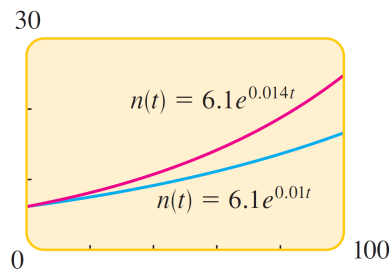
$$n(t) = 6.1e^{0.01t}$$

and find

$$n(50) = 6.1e^{0.01(50)} = 6.1e^{0.5} \approx 10.1$$

The estimated population in the year 2050 is about 10.1 billion.

The graphs in the Figure below show that a small change in the relative rate of growth will, over time, make a large difference in population size.



EXAMPLE: A certain breed of rabbit was introduced onto a small island about 8 years ago. The current rabbit population on the island is estimated to be 4100, with a relative growth rate of 55% per year.

(a) What was the initial size of the rabbit population?

(b) Estimate the population 12 years from now.

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(a) What was the initial size of the rabbit population?

(b) Estimate the population 12 years from now.

Solution:

(a) From the exponential growth model, we have

$$n(t) = n_0 e^{0.55t}$$

and we know that the population at time $t = 8$ is $n(8) = 4100$. We substitute what we know into the equation and solve for n_0 :

$$4100 = n_0 e^{0.55(8)} \implies n_0 = \frac{4100}{e^{0.55(8)}} \approx \frac{4100}{81.45} \approx 50$$

Thus, we estimate that 50 rabbits were introduced onto the island.

(b) Now that we know n_0 , we can write a formula for population growth:

$$n(t) = 50e^{0.55t}$$

Twelve years from now, $t = 8 + 12 = 20$ and

$$n(20) = 50e^{0.55(20)} \approx 2,993,707$$

We estimate that the rabbit population on the island 12 years from now will be about 3 million.

REMARK: Can the rabbit population in the Example above actually reach such a high number? In reality, as the island becomes overpopulated with rabbits, the rabbit population growth will be slowed due to food shortage and other factors. One model that takes into account such factors is the *logistic growth model*.

EXAMPLE: The population of the world in 2000 was 6.1 billion, and the estimated relative growth rate was 1.4% per year. If the population continues to grow at this rate, when will it reach 122 billion?

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Solution: We use the population growth function with $n_0 = 6.1$ billion, $r = 0.014$, and $n(t) = 122$ billion. This leads to an exponential equation, which we solve for t .

$$6.1e^{0.014t} = 122$$

$$e^{0.014t} = 20$$

$$[\ln e^{0.014t} = \ln 20]$$

$$[0.014t \ln e = \ln 20]$$

$$0.014t = \ln 20$$

$$t = \frac{\ln 20}{0.014} \approx 213.98$$

Thus, the population will reach 122 billion in approximately 214 years, that is, in the year $2000 + 214 = 2214$.

EXAMPLE: A culture starts with 10,000 bacteria, and the number doubles every 40 min.

- (a) Find a function that models the number of bacteria at time t .
- (b) Find the number of bacteria after one hour.
- (c) After how many minutes will there be 50,000 bacteria?
- (d) Sketch a graph of the number of bacteria at time t .

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Solution:

(a) To find the function that models this population growth, we need to find the rate r . To do this, we use the formula for population growth with $n_0 = 10,000$, $t = 40$, and $n(t) = 20,000$, and then solve for r .

$$10,000e^{r(40)} = 20,000$$

$$e^{40r} = 2$$

$$[\ln e^{40r} = \ln 2]$$

$$[40r \ln e = \ln 2]$$

$$40r = \ln 2$$

$$r = \frac{\ln 2}{40} \approx 0.01733$$

Now that we know $r \approx 0.01733$, we can write the function for the population growth:

$$n(t) = 10,000e^{0.01733t}$$

(b) Using the function we found in part (a) with $t = 60$ min (one hour), we get

$$n(60) = 10,000e^{0.01733(60)} \approx 28,287$$

Thus, the number of bacteria after one hour is approximately 28,000.

(c) We use the function we found in part (a) with $n(t) = 50,000$ and solve the resulting exponential equation for t .

$$10,000e^{0.01733t} = 50,000$$

$$e^{0.01733t} = 5$$

$$[\ln e^{0.01733t} = \ln 5]$$

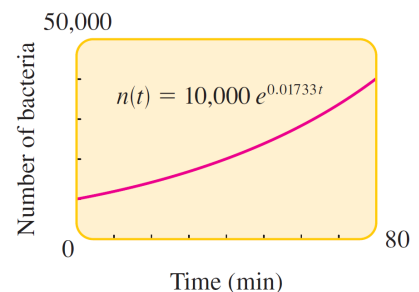
$$[0.01733t \ln e = \ln 5]$$

$$0.01733t = \ln 5$$

$$t = \frac{\ln 5}{0.01733} \approx 92.9$$

The bacterium count will reach 50,000 in approximately 93 min.

(d) The graph of the function $n(t) = 10,000e^{0.01733t}$ is shown.



Radioactive Decay

Radioactive substances decay by spontaneously emitting radiation. The rate of decay is directly proportional to the mass of the substance. This is analogous to population growth, except that the mass of radioactive material *decreases*. It can be shown that the mass $m(t)$ remaining at time t is modeled by the function

$$m(t) = m_0 e^{-rt}$$

where r is the rate of decay expressed as a proportion of the mass and m_0 is the initial mass. Physicists express the rate of decay in terms of **half-life**, the time required for half the mass to decay. We can obtain the rate r from this as follows. If h is the half-life, then a mass of 1 unit becomes $\frac{1}{2}$ unit when $t = h$. Substituting this into the model, we get

$$\begin{aligned}\frac{1}{2} &= 1 \cdot e^{-rh} \\ \ln\left(\frac{1}{2}\right) &= -rh \\ r &= -\frac{1}{h} \ln(2^{-1}) \\ r &= \frac{\ln 2}{h}\end{aligned}$$

This last equation allows us to find the rate r from the half-life h .

RADIOACTIVE DECAY MODEL: If m_0 is the initial mass of a radioactive substance with half-life h , then the mass remaining at time t is modeled by the function

$$m(t) = m_0 e^{-rt}$$

where $r = \frac{\ln 2}{h}$.

EXAMPLE: Polonium-210 (^{210}Po) has a half-life of 140 days. Suppose a sample of this substance has a mass of 300 mg.

- Find a function that models the amount of the sample remaining at time t .
- Find the mass remaining after one year.
- How long will it take for the sample to decay to a mass of 200 mg?
- Draw a graph of the sample mass as a function of time.

Solution:

- Using the model for radioactive decay with $m_0 = 300$ and

$$r = \frac{\ln 2}{140} \approx 0.00495$$

we have

$$m(t) = 300e^{-0.00495t}$$

(b) We use the function we found in part (a) with $t = 365$ (one year).

$$m(365) = 300e^{-0.00495(365)} \approx 49.256$$

Thus, approximately 49 mg of ^{210}Po remains after one year.

(c) We use the function we found in part (a) with and solve the resulting exponential equation for t .

$$300e^{-0.00495t} = 200$$

$$e^{-0.00495t} = \frac{2}{3}$$

$$\left[\ln e^{-0.00495t} = \ln \frac{2}{3} \right]$$

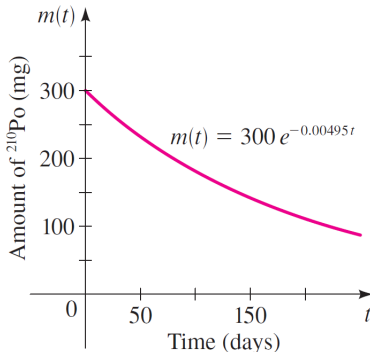
$$\left[-0.00495t \ln e = \ln \frac{2}{3} \right]$$

$$-0.00495t = \ln \frac{2}{3}$$

$$t = -\frac{\ln \frac{2}{3}}{0.00495} \approx 81.9$$

The time required for the sample to decay to 200 mg is about 82 days.

(d) A graph of the function $m(t) = 300e^{-0.00495t}$ is shown in the Figure below.



EXAMPLE: The half-life of radium-226 ($^{226}_{88}\text{Ra}$) is 1590 years.

(a) A sample of radium-226 has a mass of 100 mg. Find the formula for the mass of $^{226}_{88}\text{Ra}$ that remains after t years.

(b) Find the mass after 1000 years correct to the nearest milligram.

(c) When will the mass be reduced to 30 mg?

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(c) When will the mass be reduced to 30 mg?

Solution:

(a) Let $m(t)$ be the mass of radium-226 (in milligrams) that remains after t years. By the Theorem above we have

$$m(t) = m(0)e^{-rt}$$

where $m(0) = 100$ and

$$r = \frac{\ln 2}{1590}$$

Therefore

$$m(t) = 100e^{-(\ln 2)t/1590}$$

(b) The mass after 1000 years is

$$m(1000) = 100e^{-(\ln 2)1000/1590} \approx 65 \text{ mg}$$

(c) We use the function we found in part (a) with and solve the resulting exponential equation for t .

$$100e^{-(\ln 2)t/1590} = 30$$

$$e^{-(\ln 2)t/1590} = 0.3$$

$$\left[\ln e^{-(\ln 2)t/1590} = \ln 0.3 \right]$$

$$\left[-\frac{t \ln 2}{1590} \ln e = \ln 0.3 \right]$$

$$-\frac{t \ln 2}{1590} = \ln 0.3$$

$$t = -\frac{1590 \ln 0.3}{\ln 2} \approx 2762 \text{ years}$$

Wikipedia: Carbon-14 or radiocarbon, is a radioactive isotope of carbon discovered on February 27, 1940, by Martin Kamen and Samuel Ruben at the University of California Radiation Laboratory in Berkeley, though its existence had been suggested already in 1934 by Franz Kurie. Radiocarbon dating is a radiometric dating method that uses Carbon-14 to determine the age of carbonaceous materials up to about 60,000 years old. The technique was developed by Willard Libby and his colleagues in 1949 during his tenure as a professor at the University of Chicago. Libby estimated that the radioactivity of exchangeable carbon-14 would be about 14 disintegrations per minute (dpm) per gram. In 1960, he was awarded the Nobel Prize in chemistry for this work. One of the frequent uses of the technique is to date organic remains from archaeological sites.



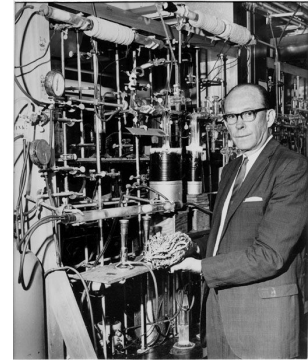
Franz Kurie



Martin Kamen



Samuel Ruben



Willard Libby

EXAMPLE: You find a skull in a nearby Native American ancient burial site and with the help of a spectrometer, discover that the skull contains 9% of the C-14 found in a modern skull. Assuming that the half life of C-14 (radiocarbon) is 5730 years, how old is the skull?

EXAMPLE: You find a skull in a nearby Native American ancient burial site and with the help of a spectrometer, discover that the skull contains 9% of the C-14 found in a modern skull. Assuming that the half life of C-14 (radiocarbon) is 5730 years, how old is the skull?

Solution: (a) Let $C(t)$ be the mass of C-14 that remains after t years. By the Theorem above we have

$$C(t) = C(0)e^{-rt}$$

where

$$r = \frac{\ln 2}{5730} \approx 0.000121$$

Now we use the fact that there is 9% remaining today to give

$$0.09C(0) = C(0)e^{-0.000121t} \implies 0.09 = e^{-0.000121t}$$

From this it follows that

$$-0.000121t = \ln 0.09 \implies t = \frac{\ln 0.09}{-0.000121} \approx 19,905$$

So, the skull is about 20,000 years old.

Newton's Law of Cooling

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that the temperature difference is not too large. Using calculus, the following model can be deduced from this law.

NEWTON'S LAW OF COOLING: If D_0 is the initial temperature difference between an object and its surroundings, and if its surroundings have temperature T_s , then the temperature of the object at time t is modeled by the function

$$T(t) = T_s + D_0e^{-kt}$$

where k is a positive constant that depends on the type of object.

EXAMPLE: A cup of coffee has a temperature of 200°F and is placed in a room that has a temperature of 70° F. After 10 min the temperature of the coffee is 150°F.

- Find a function that models the temperature of the coffee at time t .
- Find the temperature of the coffee after 15 min.
- When will the coffee have cooled to 100°F?
- Illustrate by drawing a graph of the temperature function.

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- Find a function that models the temperature of the coffee at time t .
- Find the temperature of the coffee after 15 min.
- When will the coffee have cooled to 100°F ?
- Illustrate by drawing a graph of the temperature function.

Solution:

- The temperature of the room is $T_s = 70^\circ\text{F}$, and the initial temperature difference is

$$D_0 = 200 - 70 = 130^\circ\text{F}$$

So, by Newton's Law of Cooling, the temperature after t minutes is modeled by the function

$$T(t) = 70 + 130e^{-kt}$$

We need to find the constant k associated with this cup of coffee. To do this, we use the fact that when $t = 10$, the temperature is $T(10) = 150$. So we have

$$\begin{aligned} 70 + 130e^{-10k} &= 150 \\ 130e^{-10k} &= 80 \\ e^{-10k} &= \frac{8}{13} \\ -10k &= \ln \frac{8}{13} \\ k &= -\frac{1}{10} \ln \frac{8}{13} \approx 0.04855 \end{aligned}$$

Substituting this value of k into the expression for $T(t)$, we get

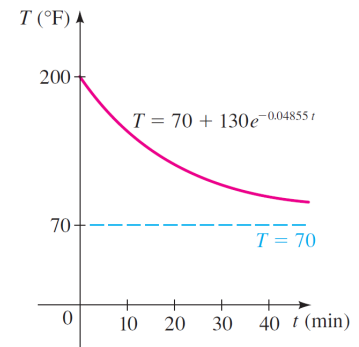
$$T(t) = 70 + 130e^{-0.04855t}$$

- We use the function we found in part (a) with $t = 15$.

$$T(15) = 70 + 130e^{-0.04855(15)} \approx 133^\circ\text{F}$$

- We use the function we found in part (a) with $T(t) = 100$ and solve the resulting exponential equation for t .

$$\begin{aligned} 70 + 130e^{-0.04855t} &= 100 \\ 130e^{-0.04855t} &= 30 \\ e^{-0.04855t} &= \frac{3}{13} \\ -0.04855t &= \ln \frac{3}{13} \\ t &= \frac{\ln \frac{3}{13}}{-0.04855} \approx 30.2 \end{aligned}$$



The coffee will have cooled to 100°F after about half an hour.

- The graph of the temperature function is shown. Notice that the line $t = 70$ is a horizontal asymptote.

Logarithmic Scales

When a physical quantity varies over a very large range, it is often convenient to take its logarithm in order to have a more manageable set of numbers.

THE RICHTER SCALE: In 1935 the American geologist Charles Richter (1900-1984) defined the magnitude M of an earthquake to be

$$M = \log \frac{I}{S}$$

where I is the intensity of the earthquake (measured by the amplitude of a seismograph reading taken 100 km from the epicenter of the earthquake) and S is the intensity of a “standard” earthquake (whose amplitude is 1 micron = 10^{-4} cm). The magnitude of a standard earthquake is

$$M = \log \frac{S}{S} = \log 1 = 0$$

Richter studied many earthquakes that occurred between 1900 and 1950. The largest had magnitude 8.9 on the Richter scale, and the smallest had magnitude 0. This corresponds to a ratio of intensities of 800,000,000, so the Richter scale provides more manageable numbers to work with. For instance, an earthquake of magnitude 6 is ten times stronger than an earthquake of magnitude 5.

EXAMPLE: The 1906 earthquake in San Francisco had an estimated magnitude of 8.3 on the Richter scale. In the same year a powerful earthquake occurred on the Colombia-Ecuador border and was four times as intense. What was the magnitude of the Colombia-Ecuador earthquake on the Richter scale?

Solution: If I is the intensity of the San Francisco earthquake, then from the definition of magnitude we have

$$M = \log \frac{I}{S} = 8.3$$

The intensity of the Colombia-Ecuador earthquake was $4I$, so its magnitude was

$$M = \log \frac{4I}{S} = \log 4 + \log \frac{I}{S} = \log 4 + 8.3 \approx 8.9$$

EXAMPLE: The 1989 Loma Prieta earthquake that shook San Francisco had a magnitude of 7.1 on the Richter scale. How many times more intense was the 1906 earthquake (see the previous Example) than the 1989 event?

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Solution: If I_1 and I_2 are the intensities of the 1906 and 1989 earthquakes, then we are required to find I_1/I_2 . To relate this to the definition of magnitude, we divide numerator and denominator by S .

$$\begin{aligned}\log \frac{I_1}{I_2} &= \log \frac{I_1/S}{I_2/S} \\ &= \log \frac{I_1}{S} - \log \frac{I_2}{S} \\ &= 8.3 - 7.1 = 1.2\end{aligned}$$

Therefore

$$\frac{I_1}{I_2} = 10^{\log(I_1/I_2)} = 10^{1.2} \approx 16$$

The 1906 earthquake was about 16 times as intense as the 1989 earthquake.

Appendix

EXAMPLE: Use the fact that the world population was 2560 million people in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century. (Assume that the growth rate is proportional to the population size.) What is the relative growth rate r ? Use the model to estimate the world population in 1993 and to predict the population in the year 2020.

Solution: Since the growth rate is proportional to the population size, we have

$$P(t) = P(0)e^{rt} \tag{1}$$

To find k we note that $P(0) = 2560$ and $P(10) = 3040$. Using this in (1) with $t = 10$, we get

$$P(10) = P(0)e^{rt} \implies 3040 = 2560e^{r \cdot 10} \implies \frac{3040}{2560} = e^{r \cdot 10}$$

hence

$$10r = \ln \frac{3040}{2560} \implies r = \frac{1}{10} \ln \frac{3040}{2560} \approx 0.017185$$

Substituting this into (1) and keeping in mind that $P(0) = 2560$, we obtain

$$P(t) = 2560e^{0.017185t} \tag{2}$$

Equation (2) helps us to estimate the world population in 1993 and to predict the population in the year 2020. In fact, the world population in 1993 was

$$P(43) = 2560e^{0.017185 \cdot 43} \approx 5360 \text{ million (actual is 5522 million)}$$

Similarly, the world population in 2020 will be

$$P(70) = 2560e^{0.017185 \cdot 70} \approx 8524 \text{ million}$$

EXAMPLE: At the start of an experiment, there are 100 bacteria. If the bacteria follow an exponential growth pattern with rate $r = 0.02$, what will be the population after 5 hours? How long will it take for the population to double?

Solution: By the Theorem above we get

$$P(t) = P(0)e^{rt}$$

where $P(0) = 100$ and $r = 0.02$. Therefore

$$P(5) = 100e^{0.02 \cdot 5} \approx 110.517 \approx 110 \text{ bacteria}$$

The second question suggests the following equation

$$2 \cdot 100 = 100e^{0.02t} \implies 2 = e^{0.02t}$$

hence

$$0.02t = \ln 2 \implies t = \frac{\ln 2}{0.02} \approx 34.6574 \text{ hours}$$

EXAMPLE: Suppose that the population of a colony of bacteria increases exponentially. At the start of an experiment, there are 6,000 bacteria, and one hour later, the population has increased to 6,400. How long will it take for the population to reach 10,000? Round your answer to the nearest hour.

Solution: By the Theorem above we get

$$P(t) = P(0)e^{rt}$$

where $P(0) = 6000$, $P(1) = 6400$ and $t = 1$. Therefore

$$6400 = 6000e^{r \cdot 1} \implies \frac{6400}{6000} = e^r \implies r = \ln\left(\frac{6400}{6000}\right) \approx 0.06454$$

so

$$P(t) = 6000e^{0.06454t}$$

The second question suggests the following equation

$$10000 = 6000e^{0.06454t} \implies \frac{10000}{6000} = e^{0.06454t}$$

hence

$$0.06454t = \ln\left(\frac{10000}{6000}\right) \implies t = \frac{1}{0.06454} \ln\left(\frac{10000}{6000}\right) \approx 7.915 \approx 8 \text{ hours}$$