Section 4.4 Laws of Logarithms

LAWS OF LOGARITHMS: If x and y are positive numbers, then

1. $\log_a(xy) = \log_a x + \log_a y$.
2. $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$.
3. $\log_a(x^r) = r\log_a x$ where r is any real number.

EXAMPLES:

1. Use the laws of logarithms to evaluate $\log_2 4$.

Solution: We have
$$\log_2 4 = \log_2 2^2 = 2\log_2 2 = 2\cdot 1 = 2$$

2. Use the laws of logarithms to evaluate $\log 1,000,000$.

Solution: We have
$$\log 1,000,000 = \log 10^6 = 6\log 10 = 6\cdot 1 = 6$$

3. Use the laws of logarithms to evaluate $\log_7\sqrt{7}$.

Solution: We have
$$\log_7\sqrt{7} = \log_7 7^{1/2} = \frac{1}{2}\log_7 7 = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

4. Use the laws of logarithms to evaluate $\log_5\dfrac{1}{\sqrt[3]{5}}$.

Solution: We have
$$\log_5\frac{1}{\sqrt[3]{5}} = \log_5\frac{1}{5^{1/3}} = \log_5 5^{-1/3} = -\frac{1}{3}\log_5 5 = -\frac{1}{3}\cdot 1 = -\frac{1}{3}$$
or
$$\log_5\frac{1}{\sqrt[3]{5}} = \log_5\frac{1}{5^{1/3}} = \log_5 1 - \log_5 5^{1/3} = 0 - \frac{1}{3}\log_5 5 = -\frac{1}{3}\log_5 5 = -\frac{1}{3}\cdot 1 = -\frac{1}{3}$$

5. Use the laws of logarithms to evaluate $\log_4 8$.

Solution: We have
$$\log_4 8 = \log_4 2^3 = \log_4(4^{1/2})^3 = \log_4 4^{(1/2)\cdot 3} = \log_4 4^{3/2} = \frac{3}{2}\log_4 4 = \frac{3}{2}\cdot 1 = \frac{3}{2}$$

6. Use the laws of logarithms to evaluate $\log_3 270 - \log_3 10$.

Solution: We have
$$\log_3 270 - \log_3 10 = \log_3\left(\frac{270}{10}\right) = \log_3 27 = \log_3 3^3 = 3\log_3 3 = 3\cdot 1 = 3$$

7. Use the laws of logarithms to evaluate $\log_2 12 + \log_2 3 - \log_2 9$.

7. Use the laws of logarithms to evaluate $\log_2 12 + \log_2 3 - \log_2 9$.

Solution: We have

$$\begin{aligned}\log_2 12 + \log_2 3 - \log_2 9 = \log_2(12 \cdot 3) - \log_2 9 = \log_2\left(\frac{12 \cdot 3}{9}\right) &= \log_2 4 = \log_2 2^2 \\ &= 2\log_2 2 = 2 \cdot 1 = 2\end{aligned}$$

EXAMPLES:

1. $\ln\left(x(x+1)\right) = \ln x + \ln(x+1)$
2. $\ln\left(\dfrac{x}{x+2}\right) = \ln x - \ln(x+2)$
3. $\ln\left(\dfrac{x(x+1)}{x+2}\right) = \ln\Big(x(x+1)\Big) - \ln(x+2) = \ln x + \ln(x+1) - \ln(x+2)$
4. $$\begin{aligned}\ln\left(\frac{x(x+1)}{(x+2)(x+3)}\right) &= \ln\Big(x(x+1)\Big) - \ln\Big((x+2)(x+3)\Big) \\ &= \Big(\ln x + \ln(x+1)\Big) - \Big(\ln(x+2) + \ln(x+3)\Big) \\ &= \ln x + \ln(x+1) - \ln(x+2) - \ln(x+3)\end{aligned}$$
5. $$\begin{aligned}\ln\left(\frac{x\sqrt{x+1}}{\sqrt[3]{x+2}(x+3)^5}\right) &= \ln\left(\frac{x(x+1)^{1/2}}{(x+2)^{1/3}(x+3)^5}\right) \\ &= \ln\Big(x(x+1)^{1/2}\Big) - \ln\Big((x+2)^{1/3}(x+3)^5\Big) \\ &= \Big(\ln x + \ln(x+1)^{1/2}\Big) - \Big(\ln(x+2)^{1/3} + \ln(x+3)^5\Big) \\ &= \ln x + \ln(x+1)^{1/2} - \ln(x+2)^{1/3} - \ln(x+3)^5 \\ &= \ln x + \frac{1}{2}\ln(x+1) - \frac{1}{3}\ln(x+2) - 5\ln(x+3)\end{aligned}$$
6. $$\begin{aligned}\ln\left(\frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}\right) = \ln\left(\frac{x^2(7x-14)^{1/3}}{(1+x^2)^4}\right) &= \ln\Big(x^2(7x-14)^{1/3}\Big) - \ln(1+x^2)^4 \\ &= \ln x^2 + \ln(7x-14)^{1/3} - \ln(1+x^2)^4 \\ &= 2\ln x + \frac{1}{3}\ln(7x-14) - 4\ln(1+x^2)\end{aligned}$$
7. $\ln\left(\dfrac{\sqrt[3]{x^2-8}\sqrt{x^3+1}}{\sqrt{1-x}(x+2)^{-3}(x^6-7x+5)^5}\right) =$

7. $\ln\left(\frac{\sqrt[3]{x^2-8}\sqrt{x^3+1}}{\sqrt{1-x}(x+2)^{-3}(x^6-7x+5)^5}\right) = \ln\left(\frac{(x^2-8)^{1/3}(x^3+1)^{1/2}}{(1-x)^{1/2}(x+2)^{-3}(x^6-7x+5)^5}\right)$

$= \ln\left((x^2-8)^{1/3}(x^3+1)^{1/2}\right) - \ln\left((1-x)^{1/2}(x+2)^{-3}(x^6-7x+5)^5\right)$

$= \left(\ln(x^2-8)^{1/3} + \ln(x^3+1)^{1/2}\right) - \left(\ln(1-x)^{1/2} + \ln(x+2)^{-3} + \ln(x^6-7x+5)^5\right)$

$= \ln(x^2-8)^{1/3} + \ln(x^3+1)^{1/2} - \ln(1-x)^{1/2} - \ln(x+2)^{-3} - \ln(x^6-7x+5)^5$

$= \frac{1}{3}\ln(x^2-8) + \frac{1}{2}\ln(x^3+1) - \frac{1}{2}\ln(1-x) + 3\ln(x+2) - 5\ln(x^6-7x+5)$

Change of Base

IMPORTANT FORMULA: For any positive a and b ($a, b \neq 1$) we have

$$\boxed{\log_b x = \frac{\log_a x}{\log_a b}}$$

In particular, if $a = e$ or 10, then

$$\boxed{\log_b x = \frac{\ln x}{\ln b} = \frac{\log x}{\log b}}$$

EXAMPLES:

1. $\log_5 4 = \frac{\ln 4}{\ln 5} = \frac{\log 4}{\log 5} \approx 0.86135$

2. $\log_4 8 = \log_{2^2} 2^3 = \frac{\log 2^3}{\log 2^2} = \frac{3\log 2}{2\log 2} = \frac{3}{2}$

3. $\log_{27}\left(\frac{1}{9}\right) = \log_{3^3}\left(\frac{1}{3^2}\right) = \log_{3^3} 3^{-2} = \frac{\log 3^{-2}}{\log 3^3} = \frac{-2\log 3}{3\log 3} = -\frac{2}{3}$

or

$$\log_{27}\left(\frac{1}{9}\right) = \log_{3^3}\left(\frac{1}{3^2}\right) = \log_{3^3} 1 - \log_{3^3} 3^2 = 0 - \log_{3^3} 3^2 = -\frac{\log 3^2}{\log 3^3} = -\frac{2\log 3}{3\log 3} = -\frac{2}{3}$$

4. $\log_9\left(\frac{1}{\sqrt{27}}\right) = \log_9\left(\frac{1}{27^{1/2}}\right) = \log_{3^2}\left(\frac{1}{(3^3)^{1/2}}\right) = \log_{3^2}\left(\frac{1}{3^{3/2}}\right)$

Now we can proceed in two different ways. Either

$$\log_{3^2}\left(\frac{1}{3^{3/2}}\right) = \log_{3^2} 3^{-3/2} = \frac{\log 3^{-3/2}}{\log 3^2} = \frac{(-3/2)\log 3}{2\log 3} = \frac{-3/2}{2} = -\frac{3}{4}$$

or

$$\log_{3^2}\left(\frac{1}{3^{3/2}}\right) = \log_{3^2} 1 - \log_{3^2} 3^{3/2} = 0 - \log_{3^2} 3^{3/2} = -\log_{3^2} 3^{3/2} = -\frac{\log 3^{3/2}}{\log 3^2}$$

$$= -\frac{(3/2)\log 3}{2\log 3}$$

$$= -\frac{3/2}{2} = -\frac{3}{4}$$