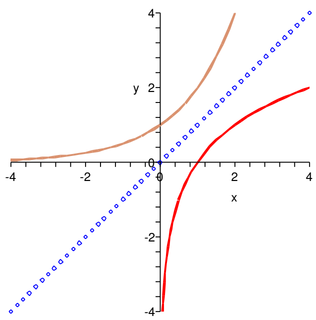
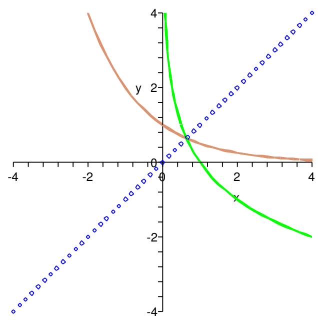
Section 4.3 Logarithmic Functions

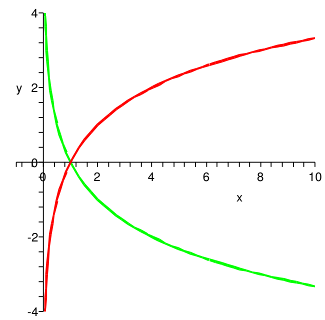
DEFINITION: Let a be a positive number with $a \neq 1$. The **logarithmic function with base** $\boldsymbol{a}$, denoted by $\mathbf{log}_{\boldsymbol{a}}$, is defined by

$$\boxed{\log_a x = y \quad \Longleftrightarrow \quad a^y = x}$$

So, $\log_a x$ is the *exponent* to which the base a must be raised to give x.







BASIC PROPERTIES: $f(x) = \log_a x$ is a continuous function with domain $(0, \infty)$ and range $(-\infty, \infty)$. Moreover,

$$\boxed{\log_a(a^x) = x \text{ for every } x \in \mathbb{R}, \quad a^{\log_a x} = x \text{ for every } x > 0}$$

REMARK: It immediately follows from property 1 that

$$\boxed{\log_a a = 1, \quad \log_a 1 = 0}$$

BASIC CALCULUS PROPERTIES:

1. If $a > 1$, then $\log_a x \to \infty$ as $x \to \infty$ and $\log_a x \to -\infty$ as $x \to 0^+$.

2. If $0 < a < 1$, then $\log_a x \to -\infty$ as $x \to \infty$ and $\log_a x \to \infty$ as $x \to 0^+$.

EXAMPLES:

1. $\log_2 2 = 1, \ \log_2 4 = 2, \ \log_2 8 = 3, \ \log_2 16 = 4$

2. $\log_3 3 = 1, \ \log_3 9 = 2, \ \log_3 27 = 3, \ \log_3 81 = 4$

3. $\log_3\left(\dfrac{1}{3}\right) = \log_3 3^{-1} = -1, \ \log_3\left(\dfrac{1}{9}\right) = \log_3 3^{-2} = -2, \ \log_3\left(\dfrac{1}{27}\right) = \log_3 3^{-3} = -3$

4. $\log_5 \sqrt{5} = \log_5 5^{1/2} = \dfrac{1}{2}, \ \log_7 \sqrt[3]{7} = \log_7 7^{1/3} = \dfrac{1}{3}$

5. $\log_{11}\left(\dfrac{1}{\sqrt[5]{11}}\right) = \log_{11}\left(\dfrac{1}{11^{1/5}}\right) = \log_{11} 11^{-1/5} = -\dfrac{1}{5}$

6. $\log_4 8 = \log_4 2^3 = \log_4 (4^{1/2})^3 = \log_4 4^{(1/2)\cdot 3} = \log_4 4^{3/2} = \dfrac{3}{2}$

7. $\log_{27}\left(\dfrac{1}{9}\right) = \log_{27}\left(\dfrac{1}{3^2}\right) = \log_{27} 3^{-2} = \log_{27}(27^{1/3})^{-2} = \log_{27} 27^{-2/3} = -\dfrac{2}{3}$

8. $\log_9\left(\dfrac{1}{\sqrt{27}}\right) = \log_9\left(\dfrac{1}{27^{1/2}}\right) = \log_9 27^{-1/2} = \log_9 (3^3)^{-1/2} = \log_9 3^{-3/2} = \log_9 (9^{1/2})^{-3/2}$

$$= \log_9 9^{-3/4} = -\frac{3}{4}$$

9. $\log_2 3 \approx 1.58496$, $\log_3 5 \approx 1.46497$, $\log_7 1000 \approx 3.54988$

Common and Natural Logarithms

DEFINITION: The logarithm with base e is called the **natural logarithm** and has a special notation:

$$\boxed{\log_e x = \ln x}$$

BASIC PROPERTIES:

1. $\ln(e^x) = x$ for every $x \in \mathbb{R}$.

2. $e^{\ln x} = x$ for every $x > 0$.

REMARK: It immediately follows from property 1 that $\ln e = 1$.

EXAMPLES: $\ln e^2 = 2$, $\ln \sqrt[3]{e} = \ln(e^{1/3}) = \dfrac{1}{3}$, $\ln(1/e) = \ln(e^{-1}) = -1$, $\ln 3 \approx 1.09861$

DEFINITION: The logarithm with base 10 is called the **common logarithm** and has a special notation:

$$\boxed{\log_{10} x = \log x}$$

BASIC PROPERTIES:

1. $\log(10^x) = x$ for every $x \in \mathbb{R}$.

2. $10^{\log x} = x$ for every $x > 0$.

REMARK: It immediately follows from property 1 that $\log 10 = 1$.

EXAMPLES:

1. $\log 100 = \log 10^2 = 2$, $\log 1{,}000{,}000{,}000 = \log 10^9 = 9$, $\log 10^{100} = 100$

2. $\log 0.01 = \log 10^{-2} = -2$, $\log \sqrt{10} = \log 10^{1/2} = \dfrac{1}{2}$, $\log \dfrac{1}{\sqrt[7]{10}} = \log \dfrac{1}{10^{1/7}} = \log 10^{-1/7} = -\dfrac{1}{7}$

3. $\log 3 \approx 0.47712$, $\log 15 \approx 1.17609$, $\log 101 \approx 2.00432$, $\log 0.2 \approx -0.69897$