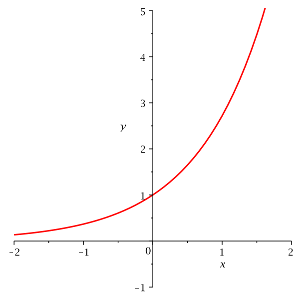
Section 4.2 The Natural Exponential Function

It is known that

$$\left(1+\frac{1}{x}\right)^x \to 2.71828182845904523536028747135266249775724709369...$$

as $x \to \pm\infty$. We denote this number by e.

x	$\left(1+\frac{1}{x}\right)^x$	Value
1	2^1	2
10	1.1^{10}	2.593742460
100	1.01^{100}	2.704813829
1000	1.001^{1000}	2.716923932
10000	1.0001^{10000}	2.718145927



DEFINITION: The **natural exponential function** is $f(x) = e^x$.

PROPERTIES OF THE NATURAL EXPONENTIAL FUNCTION: The exponential function $f(x) = e^x$ is a continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. Thus $e^x > 0$ for all x. Also

$$e^x \to 0 \text{ as } x \to -\infty \qquad \text{and} \qquad e^x \to \infty \text{ as } x \to \infty$$

So the x-axis is a horizontal asymptote of $f(x) = e^x$.

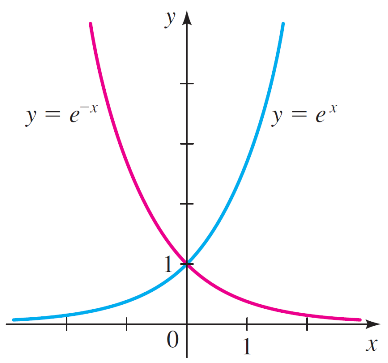
EXAMPLE: Sketch the graph of each function.

(a) $f(x) = e^{-x}$ (b) $g(x) = 3e^{0.5x}$

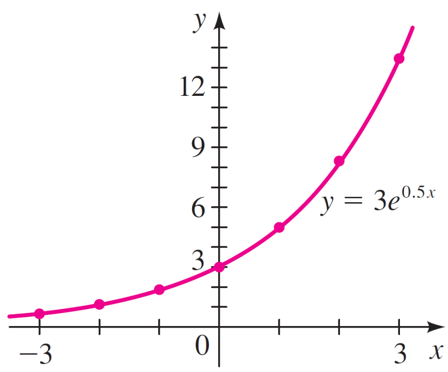
Solution:

(a) We start with the graph of $y = e^x$ and reflect in the y-axis to obtain the graph of $y = e^{-x}$.

(b) We calculate several values, plot the resulting points, then connect the points with a smooth curve.



x	$f(x) = 3e^{0.5x}$
−3	0.67
−2	1.10
−1	1.82
0	3.00
1	4.95
2	8.15
3	13.45



Continuous Compounded Interest

If \$100 is invested at 2% interest, **compounded annually**, then after 1 year the investment is worth

$$\$100(1+0.02) = \$102$$

If \$100 is invested at 2% interest, **compounded semiannually**, then after 1 year the investment is worth

$$\$100\left(1+\frac{0.02}{2}\right) = \$101 \text{ (after first 6 months)}$$

$$\$101\left(1+\frac{0.02}{2}\right) = \$102.01 \text{ (after 1 year)}$$

The same result can be obtained in a more elegant way:

$$\$100\left(1+\frac{0.02}{2}\right) = \$101 \text{ (after first 6 months)}$$

$$\$100\left(1+\frac{0.02}{2}\right)\left(1+\frac{0.02}{2}\right) = \$100\left(1+\frac{0.02}{2}\right)^2 = \$102.01 \text{ (after 1 year)}$$

If \$100 is invested at 2% interest, **compounded quarterly**, then after 1 year the investment is worth

$$\$100\left(1+\frac{0.02}{4}\right) = \$100.5 \text{ (after first 3 months)}$$

$$\$100.5\left(1+\frac{0.02}{4}\right) = \$101.0025 \text{ (after first 6 months)}$$

$$\$101.0025\left(1+\frac{0.02}{4}\right) = \$101.5075125 \text{ (after first 9 months)}$$

$$\$101.5075125\left(1+\frac{0.02}{4}\right) = \$102.0150500625 \text{ (after 1 year)}$$

As before, the same result can be obtained in a more elegant way:

$$\$100\left(1+\frac{0.02}{4}\right) = \$100.5 \text{ (after first 3 months)}$$

$$\$100\left(1+\frac{0.02}{4}\right)\left(1+\frac{0.02}{4}\right) = \$100\left(1+\frac{0.02}{4}\right)^2 = \$101.0025 \text{ (after first 6 months)}$$

$$\$100\left(1+\frac{0.02}{4}\right)^2\left(1+\frac{0.02}{4}\right) = \$100\left(1+\frac{0.02}{4}\right)^3 = \$101.5075125 \text{ (after first 9 months)}$$

$$\$100\left(1+\frac{0.02}{4}\right)^3\left(1+\frac{0.02}{4}\right) = \$100\left(1+\frac{0.02}{4}\right)^4 = \$102.0150500625 \text{ (after 1 year)}$$

In general, if we invest A_0 dollars at interest r, compounded n times a year, then after 1 year the investment is worth

$$A_0\left(1+\frac{r}{n}\right)^n \text{ dollars}$$

Moreover, after t years the investment is worth

$$\boxed{A(t) = A_0\left(1+\frac{r}{n}\right)^{nt}} \qquad (6)$$

QUESTION: What happens if $n \to \infty$?

Answer: We have

$$\begin{aligned} A(t) &= \lim_{n\to\infty} A_0 \left(1 + \frac{r}{n}\right)^{nt} = \lim_{n\to\infty} A_0 \left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} = A_0 \left[\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} \\ &= A_0 \left[\lim_{n\to\infty} \left(1 + \frac{1}{n/r}\right)^{n/r}\right]^{rt} = A_0 \left[\lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^{m}\right]^{rt} = A_0 e^{rt} \end{aligned}$$

so

$$\boxed{A(t) = A_0 e^{rt}} \tag{7}$$

EXAMPLE: If \$100 is invested at 2% interest, **compounded continuously**, then after 1 year the investment is worth

$$A(1) = \$100e^{0.02\cdot 1} \approx \$102.02$$

EXAMPLE: If \$200,000 is borrowed at 5.5% interest, find the amounts due at the end of 30 years if the interest compounded (i) annually, (ii) quarterly, (iii) monthly, (iv) continuously.

Solution:

(i) By (6) we have

$$A(30) = A_0 \left(1 + \frac{r}{n}\right)^{n\cdot 30} = \$200,000 \left(1 + \frac{0.055}{1}\right)^{1\cdot 30} \approx \$996,790.26$$

(ii) By (6) we have

$$A(30) = A_0 \left(1 + \frac{r}{n}\right)^{n\cdot 30} = \$200,000 \left(1 + \frac{0.055}{4}\right)^{4\cdot 30} \approx \$1,029,755.36$$

which gives $\approx$ \$32,965.10 difference between (ii) and (i).

(iii) By (6) we have

$$A(30) = A_0 \left(1 + \frac{r}{n}\right)^{n\cdot 30} = \$200,000 \left(1 + \frac{0.055}{12}\right)^{12\cdot 30} \approx \$1,037,477.57$$

which gives $\approx$ \$7,722.21 difference between (iii) and (ii).

(iv) By (7) we have

$$A(30) = A_0 e^{r\cdot 30} = \$200,000e^{0.055\cdot 30} \approx \$1,041,395.97$$

which gives $\approx$ \$3,918.38 difference between (iv) and (iii).

EXAMPLE: If \$200,000 is borrowed at 5.6% interest, find the amounts due at the end of 30 years if the interest compounded (i) annually, (ii) quarterly, (iii) monthly, (iv) continuously.

EXAMPLE: If \$200,000 is borrowed at 5.6% interest, find the amounts due at the end of 30 years if the interest compounded (i) annually, (ii) quarterly, (iii) monthly, (iv) continuously.

Solution:

(i) By (6) we have

$$A(30) = A_0 \left(1 + \frac{r}{n}\right)^{n \cdot 30} = \$200,000 \left(1 + \frac{0.056}{1}\right)^{1 \cdot 30} \approx \$1,025,528.05$$

which gives $\approx$ \$28,737.79 difference between 5.6% and 5.5%.

(ii) By (6) we have

$$A(30) = A_0 \left(1 + \frac{r}{n}\right)^{n \cdot 30} = \$200,000 \left(1 + \frac{0.056}{4}\right)^{4 \cdot 30} \approx \$1,060,680.53$$

which gives $\approx$ \$35,152.48 difference between (ii) and (i).

(iii) By (6) we have

$$A(30) = A_0 \left(1 + \frac{r}{n}\right)^{n \cdot 30} = \$200,000 \left(1 + \frac{0.056}{12}\right)^{12 \cdot 30} \approx \$1,068,925.95$$

which gives $\approx$ \$8,245.43 difference between (iii) and (ii).

(iv) By (7) we have

$$A(30) = A_0 e^{r \cdot 30} = \$200,000 e^{0.056 \cdot 30} \approx \$1,073,111.19$$

which gives $\approx$ \$4,185.24 difference between (iv) and (iii).