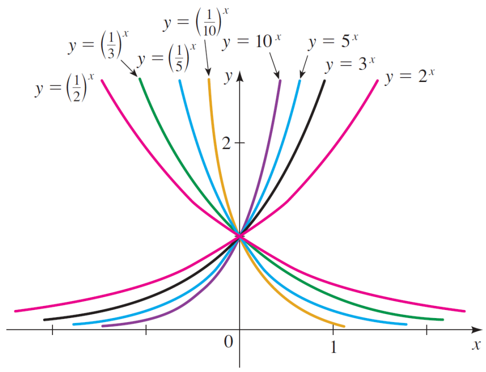
Section 4.1 Exponential Functions

DEFINITION: An **exponential function** is a function of the form

$$f(x) = a^x$$

where a is a positive constant.

x	$\left(\frac{1}{10}\right)^x$
-3	$\left(\frac{1}{10}\right)^{-3} = 10^3 = 1000$
-2	$\left(\frac{1}{10}\right)^{-2} = 10^2 = 100$
-1	$\left(\frac{1}{10}\right)^{-1} = 10^1 = 10$
0	$\left(\frac{1}{10}\right)^{0} = 1$
1	$\left(\frac{1}{10}\right)^{1} = \frac{1}{10} = 0.1$
2	$\left(\frac{1}{10}\right)^{2} = \frac{1}{10^2} = 0.01$
3	$\left(\frac{1}{10}\right)^{3} = \frac{1}{10^3} = 0.001$



x	10^x
-3	$10^{-3} = \frac{1}{10^3} = \frac{1}{1000} = 0.001$
-2	$10^{-2} = \frac{1}{10^2} = \frac{1}{100} = 0.01$
-1	$10^{-1} = \frac{1}{10} = 0.1$
0	$10^0 = 1$
1	$10^1 = 10$
2	$10^2 = 100$
3	$10^3 = 1000$

BASIC ALGEBRAIC PROPERTIES:

1. $a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}$ if n is a positive integer.

2. $a^0 = 1$.

3. $a^{-n} = \dfrac{1}{a^n}$.

4. $a^{p/q} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$.

THEOREM: If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is a continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. In particular, $a^x > 0$ for all x. If $a, b > 0$ and $x, y \in \mathbb{R}$, then

1. $a^{x+y} = a^x a^y$ 2. $a^{x-y} = \dfrac{a^x}{a^y}$ 3. $(a^x)^y = a^{xy}$ 4. $(ab)^x = a^x b^x$

BASIC CALCULUS PROPERTIES:

1. If $a > 1$, then $a^x \to \infty$ as $x \to \infty$ and $a^x \to 0$ as $x \to -\infty$.

2. If $0 < a < 1$, then $a^x \to 0$ as $x \to \infty$ and $a^x \to \infty$ as $x \to -\infty$.

REMARK: The x-axis is a horizontal asymptote of $f(x) = a^x$.

EXAMPLE: Graph the following functions:

(a) $f(x) = 2^{x-1}$

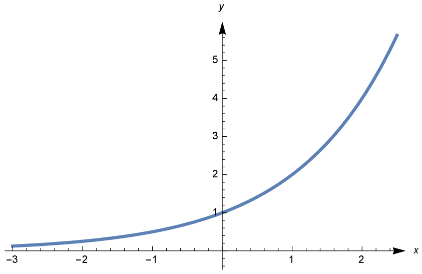
(b) $g(x) = 2^x - 1$

(c) $h(x) = -2^x$

(d) $p(x) = 2^{-x}$

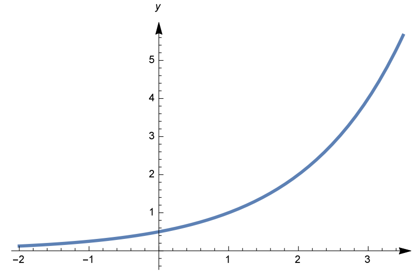
EXAMPLE: Graph the following functions:

(a) $f(x) = 2^{x-1}$



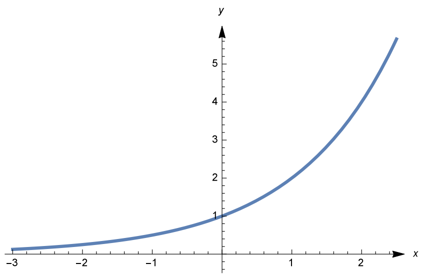
$y = 2^x$

$\Longrightarrow$



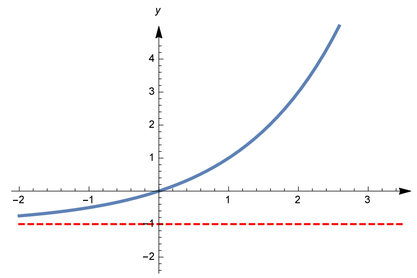
$f(x) = 2^{x-1}$ (horizontal shift)

(b) $g(x) = 2^x - 1$



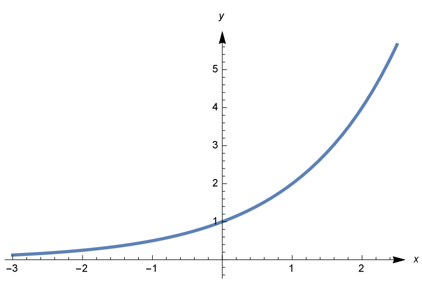
$y = 2^x$

$\Longrightarrow$



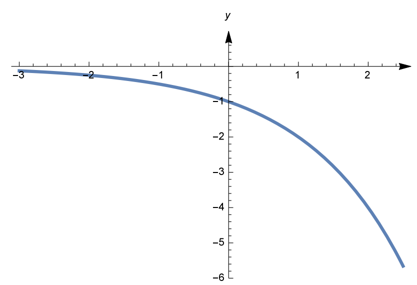
$g(x) = 2^x - 1$ (vertical shift)

(c) $h(x) = -2^x$



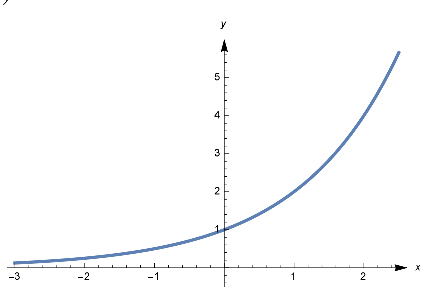
$y = 2^x$

$\Longrightarrow$



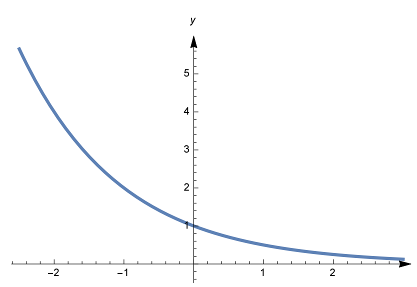
$h(x) = -2^x$ (reflection)

(d) $p(x) = 2^{-x}$



$y = 2^x$

$\Longrightarrow$



$p(x) = 2^{-x}$ (reflection)

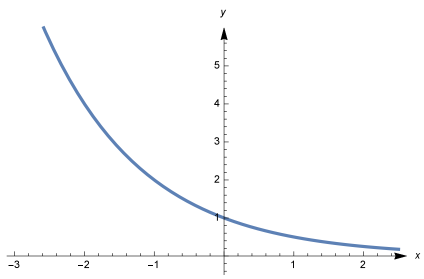
REMARK: An other way to graph $p(x)$ is to rewrite it as $p(x) = \dfrac{1}{2^x} = \left(\dfrac{1}{2}\right)^x$, which gives the same result by the Figure on page 1.

EXAMPLE: Graph $f(x) = -2^{-x}$.

Solution: Note that

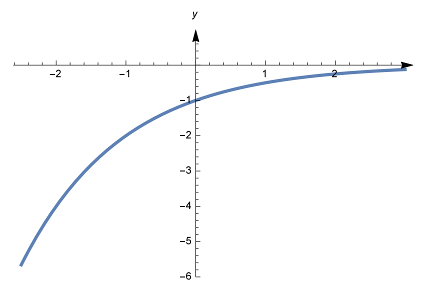
$$f(x) = -2^{-x} = -\frac{1}{2^x} = -\left(\frac{1}{2}\right)^x$$

Therefore



$$y = \left(\frac{1}{2}\right)^x$$

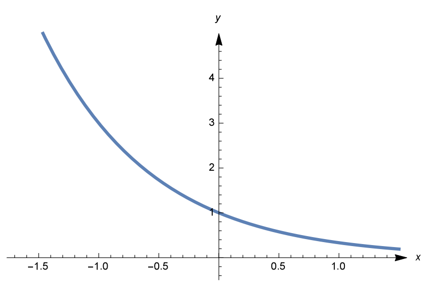
$\Longrightarrow$



$$f(x) = -\left(\frac{1}{2}\right)^x \text{ (reflection)}$$

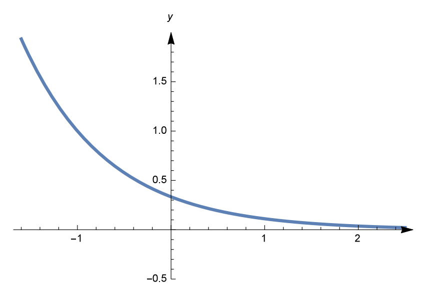
EXAMPLE: Graph $f(x) = \left(\frac{1}{3}\right)^{x+1} + 2$.

Solution: We have



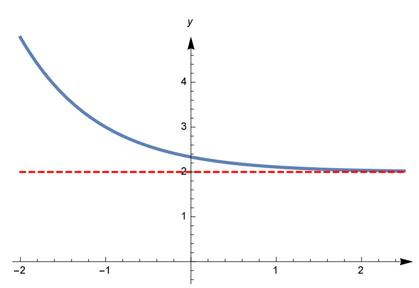
$$y = \left(\frac{1}{3}\right)^x$$

$\Longrightarrow$



$$y = \left(\frac{1}{3}\right)^{x+1} \text{ (horizontal shift)}$$

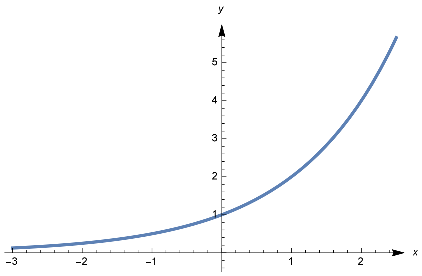
$\Downarrow$



$$f(x) = \left(\frac{1}{3}\right)^{x+1} + 2 \text{ (vertical shift)}$$

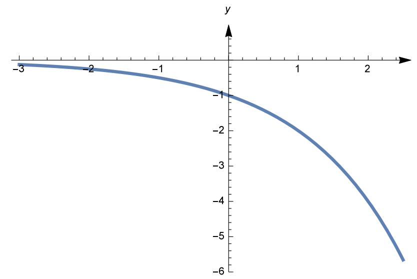
EXAMPLE: Graph $f(x) = -2^{1-x} + 3$.

Solution: We have



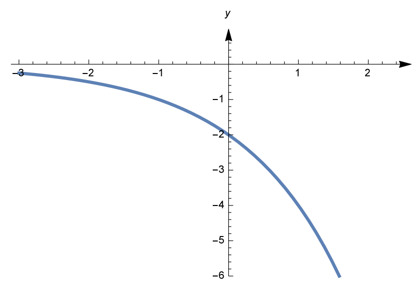
$y = 2^x$

$\Longrightarrow$



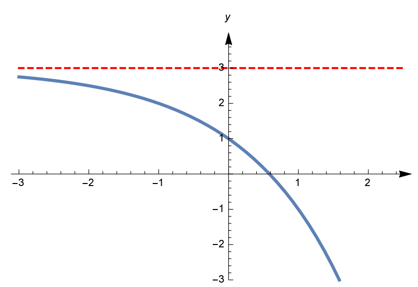
$y = -2^x$ (reflection)

$\Downarrow$



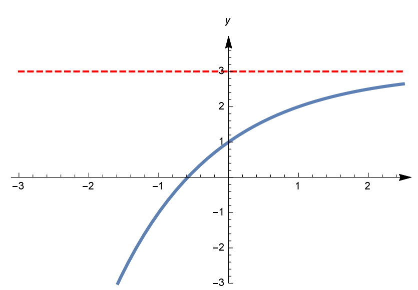
$y = -2^{x+1}$ (horizontal shift)

$\Downarrow$



$y = -2^{x+1} + 3$ (vertical shift)

$\Downarrow$



$f(x) = -2^{-x+1} + 3$ (reflection)