Section 3.7 Rational Functions

A rational function is a function of the form
\[ r(x) = \frac{P(x)}{Q(x)} \]
where \( P \) and \( Q \) are polynomials.

Rational Functions and Asymptotes

The \textit{domain} of a rational function consists of all real numbers \( x \) except those for which the denominator is zero. When graphing a rational function, we must pay special attention to the behavior of the graph near those \( x \)-values. We begin by graphing a very simple rational function.

EXAMPLE: Sketch a graph of the rational function \( f(x) = \frac{1}{x} \), and state the domain and range.

Solution: First note that the function \( f(x) = \frac{1}{x} \) is not defined for \( x = 0 \). The tables below show the behavior of \( f \) near zero.

\[
\begin{array}{|c|c|}
\hline
x & f(x) \\
\hline
-0.1 & -10 \\
-0.01 & -100 \\
-0.00001 & -100,000 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
x & f(x) \\
\hline
0.1 & 10 \\
0.01 & 100 \\
0.00001 & 100,000 \\
\hline
\end{array}
\]

This behavior can be described in the following analytical way:
\[ f(x) \to -\infty \text{ as } x \to 0^- \text{ and } f(x) \to \infty \text{ as } x \to 0^+ \]

The next two tables show how \( f(x) \) changes as \(|x|\) becomes large.

\[
\begin{array}{|c|c|}
\hline
x & f(x) \\
\hline
-10 & -0.1 \\
-100 & -0.01 \\
-100,000 & -0.00001 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
x & f(x) \\
\hline
10 & 0.1 \\
100 & 0.01 \\
100,000 & 0.00001 \\
\hline
\end{array}
\]

This behavior can be described in the following analytical way:
\[ f(x) \to 0 \text{ as } x \to -\infty \text{ and } f(x) \to 0 \text{ as } x \to \infty \]
Using the information in these tables and plotting a few additional points, we obtain the graph.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x) = \frac{1}{x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>$-\frac{1}{2}$</td>
</tr>
<tr>
<td>-1</td>
<td>$-1$</td>
</tr>
<tr>
<td>$-\frac{1}{2}$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>

The function $f$ is defined for all values of $x$ other than 0, so the domain is $\{x \mid x \neq 0\}$. From the graph we see that the range is $\{y \mid y \neq 0\}$. A pure analytical way to find the range is to find the inverse of $f$. Since $f^{-1}(x) = \frac{1}{x}$ and the domain of $f^{-1}$ is $\{x \mid x \neq 0\}$, the range of $f$ is $\{y \mid y \neq 0\}$.

In the Example above we used the following arrow notation.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x \to a^-$</td>
<td>$x$ approaches $a$ from the left</td>
</tr>
<tr>
<td>$x \to a^+$</td>
<td>$x$ approaches $a$ from the right</td>
</tr>
<tr>
<td>$x \to -\infty$</td>
<td>$x$ goes to negative infinity; that is, $x$ decreases without bound</td>
</tr>
<tr>
<td>$x \to \infty$</td>
<td>$x$ goes to infinity; that is, $x$ increases without bound</td>
</tr>
</tbody>
</table>

The line $x = 0$ is called a *vertical asymptote* of the graph of $f(x) = \frac{1}{x}$, and the line $y = 0$ is a *horizontal asymptote*. Informally speaking, an asymptote of a function is a line that the graph of the function gets closer and closer to as one travels along that line.

### Definition of Vertical and Horizontal Asymptotes

1. The line $x = a$ is a **vertical asymptote** of the function $y = f(x)$ if $y$ approaches $\pm\infty$ as $x$ approaches $a$ from the right or left.

   ![Vertical Asymptote](image)

   - $y \to \infty$ as $x \to a^+$
   - $y \to -\infty$ as $x \to a^-$

2. The line $y = b$ is a **horizontal asymptote** of the function $y = f(x)$ if $y$ approaches $b$ as $x$ approaches $\pm\infty$.

   ![Horizontal Asymptote](image)

   - $y \to b$ as $x \to \infty$
   - $y \to b$ as $x \to -\infty$
Transformations of $\frac{1}{x}$

A rational function of the form

$$r(x) = \frac{ax + b}{cx + d}$$

can be graphed by shifting, stretching, and/or reflecting the graph of $f(x) = \frac{1}{x}$ using the transformations studied in Section 2.4. (Such functions are called linear fractional transformations.)

EXAMPLE: Sketch a graph of each rational function, and state the domain and range.

(a) $r(x) = \frac{2}{x - 3}$

(b) $s(x) = \frac{3x + 5}{x + 2}$

Solution:

(a) The graph of $r(x) = \frac{2}{x - 3}$ can be obtained from the graph of $f(x) = \frac{1}{x}$ by shifting 3 units to the right and stretching vertically by a factor of 2. Thus, $r$ has vertical asymptote $x = 3$ and horizontal asymptote $y = 0$.

The function $r$ is defined for all values of $x$ other than 3, so the domain is $\{x \mid x \neq 3\}$. From the graph we see that the range is $\{y \mid y \neq 0\}$. A pure analytical way to find the range is to find the inverse of $r$. We have

**Step 1: Replace $r(x)$ by $y$:**

$$y = \frac{2}{x - 3}$$

**Step 2: Solve for $x$:**

$$y = \frac{2}{x - 3} \quad \Rightarrow \quad y(x - 3) = 2 \quad \Rightarrow \quad xy - 3y = 2 \quad \Rightarrow \quad xy = 3y + 2$$

therefore

$$x = \frac{3y + 2}{y} = \frac{3y}{y} + \frac{2}{y} = 3 + \frac{2}{y}$$

**Step 3: Replace $x$ by $r^{-1}(x)$ and $y$ by $x$:**

$$r^{-1}(x) = 3 + \frac{2}{x}$$

Since the domain of $r^{-1}$ is $\{x \mid x \neq 0\}$, the range of $r$ is $\{y \mid y \neq 0\}$. 

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(b) We have

\[
\frac{3x + 5}{x + 2} = \frac{3x + 6 - 1}{x + 2} = \frac{3(x + 2) - 1}{x + 2} = \frac{3(x + 2) - 1}{x + 2} = 3 - \frac{1}{x + 2}
\]

We can obtain the same result using either long division

\[
\begin{array}{c|cc}
  & 3 & \\
\hline
x + 2 & 3x + 5 \\
  & -3x - 6 \\
  & -1 \\
\end{array}
\]

or synthetic division

\[
\begin{array}{c|cc}
  & 3 & 5 \\
\hline
-2 & -6 \\
  & 3 & -1 \\
\end{array}
\]

It follows that the graph of \( s(x) = \frac{3x + 5}{x + 2} \) can be obtained from the graph of \( f(x) = \frac{1}{x} \) by shifting 2 units to the left, reflecting in the \( x \)-axis, and shifting upward 3 units. Thus, \( s \) has vertical asymptote \( x = -2 \) and horizontal asymptote \( y = 3 \).

The function \( s \) is defined for all values of \( x \) other than \(-2\), so the domain is \( \{x \mid x \neq -2\} \).

From the graph we see that the range is \( \{y \mid y \neq 3\} \). A pure analytical way to find the range is to find the inverse of \( s \). We have

**Step 1: Replace \( s(x) \) by \( y \):**

\[ y = \frac{3x + 5}{x + 2} \]

**Step 2: Solve for \( x \):**

\[
y = \frac{3x + 5}{x + 2} \implies y(x + 2) = 3x + 5 \implies xy + 2y = 3x + 5 \implies xy - 3x = 5 - 2y
\]

therefore

\[
x(y - 3) = 5 - 2y \implies x = \frac{5 - 2y}{y - 3}
\]

**Step 3: Replace \( x \) by \( s^{-1}(x) \) and \( y \) by \( x \):**

\[ s^{-1}(x) = \frac{5 - 2x}{x - 3} \]

Since the domain of \( s^{-1} \) is \( \{x \mid x \neq 3\} \), the range of \( s \) is \( \{y \mid y \neq 3\} \).
EXAMPLE: Sketch a graph of each rational function, and state the domain and range.

(a) \( r(x) = \frac{3}{x+1} \)  
(b) \( s(x) = \frac{x-5}{x-2} \)

Solution:

(a) The graph of \( r(x) = \frac{3}{x+1} \) can be obtained from the graph of \( f(x) = \frac{1}{x} \) by shifting 1 units to the left and stretching vertically by a factor of 3. Thus, \( r \) has vertical asymptote \( x = -1 \) and horizontal asymptote \( y = 0 \).

![Graph of r(x)](image)

The function \( r \) is defined for all values of \( x \) other than \(-1\), so the domain is \( \{ x \mid x \neq -1 \} \). From the graph we see that the range is \( \{ y \mid y \neq 0 \} \). A pure analytical way to find the range is to find the inverse of \( r \). Since \( r^{-1}(x) = -1 + \frac{3}{x} \) and the domain of \( r^{-1} \) is \( \{ x \mid x \neq 0 \} \), the range of \( r \) is \( \{ y \mid y \neq 0 \} \).

(b) We have

\[
\frac{x-5}{x-2} = \frac{x-2-3}{x-2} = \frac{x-2}{x-2} + \frac{3}{x-2} = -1 + \frac{3}{x-2}
\]

It follows that the graph of \( s(x) = -\frac{x-5}{x-2} \) can be obtained from the graph of \( f(x) = \frac{1}{x} \) by shifting 2 units to the right, stretching vertically by a factor of 3, and shifting downward 1 unit. Thus, \( s \) has vertical asymptote \( x = 2 \) and horizontal asymptote \( y = -1 \).

![Graph of s(x)](image)

The function \( r \) is defined for all values of \( x \) other than \( 2 \), so the domain is \( \{ x \mid x \neq 2 \} \). From the graph we see that the range is \( \{ y \mid y \neq -1 \} \). A pure analytical way to find the range is to find the inverse of \( s \). Since \( s^{-1}(x) = \frac{2x+5}{x+1} \) and the domain of \( s^{-1} \) is \( \{ x \mid x \neq -1 \} \), the range of \( s \) is \( \{ y \mid y \neq -1 \} \).
Asymptotes of Rational Functions

The methods of the two previous Examples work only for simple rational functions. To graph more complicated ones, we need to take a closer look at the behavior of a rational function near its vertical and horizontal asymptotes.

EXAMPLE: Graph the rational function \( r(x) = \frac{2x^2 - 4x + 5}{x^2 - 2x + 1} \), and state the domain and range.

Solution:

**Vertical Asymptote:** We first factor the denominator

\[
r(x) = \frac{2x^2 - 4x + 5}{x^2 - 2x + 1} = \frac{2x^2 - 4x + 5}{(x - 1)^2}
\]

The line \( x = 1 \) is a vertical asymptote because the denominator of \( r \) is zero and the numerator is nonzero when \( x = 1 \).

To see what the graph of \( r \) looks like near the vertical asymptote, we make tables of values for \( x \)-values to the left and to the right of 1. From the tables shown below we see that

\[
\begin{align*}
\lim_{x \to 1^-} r(x) &\to \infty \\
\lim_{x \to 1^+} r(x) &\to \infty
\end{align*}
\]

Thus, near the vertical asymptote \( x = 1 \), the graph of \( r \) has the shape shown in the Figure below.
**Horizontal Asymptote:** The horizontal asymptote is the value $y$ approaches as $x \to \pm \infty$. To help us find this value, we divide both numerator and denominator by $x^2$, the highest power of $x$ that appears in the expression:

$$y = \frac{2x^2 - 4x + 5}{x^2 - 2x + 1} = \frac{(2x^2 - 4x + 5) \cdot \frac{1}{x^2}}{(x^2 - 2x + 1) \cdot \frac{1}{x^2}} = \frac{2x^2 - 4x + 5}{x^2} = \frac{2x^2}{x^2} - \frac{4x}{x^2} + \frac{5}{x^2} = \frac{2}{1} - \frac{4}{x} + \frac{5}{x^2}$$

One can see that the fractional expressions $\frac{4}{x}$, $\frac{5}{x^2}$, $\frac{2}{x}$, and $\frac{1}{x^2}$ all approach 0 as $x \to \pm \infty$. So as $x \to \pm \infty$, we have

$$y = \frac{2 - \frac{4}{x} + \frac{5}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}} \to \frac{2 - 0 + 0}{1 - 0 + 0} = \frac{2}{1} = 2$$

Thus, the horizontal asymptote is the line $y = 2$. Since the graph must approach the horizontal asymptote, we can complete it as in the Figure below.

![Graph](image)

**Domain and Range:** The function $r$ is defined for all values of $x$ other than 1, so the domain is $\{x \mid x \neq 1\}$. From the graph we see that the range is $\{y \mid y > 2\}$.

**EXAMPLE:** Graph the rational function $r(x) = \frac{3x^2 + x + 12}{x^2 - 5x + 4}$, and state the domain and range.
EXAMPLE: Graph the rational function \( r(x) = \frac{3x^2 + x + 12}{x^2 - 5x + 4} \), and state the domain and range.

Solution:

**Vertical Asymptotes:** We first factor the denominator

\[
r(x) = \frac{3x^2 + x + 12}{x^2 - 5x + 4} = \frac{3x^2 + x + 12}{(x-1)(x-4)}
\]

The lines \( x = 1 \) and \( x = 4 \) are vertical asymptotes because the denominator of \( r \) is zero and the numerator is nonzero when \( x = 1 \) or \( x = 4 \).

**Behavior Near Vertical Asymptotes:** We need to know whether \( y \to \infty \) or \( y \to -\infty \) on each side of each vertical asymptote. To determine the sign of \( y \) for \( x \)-values near the vertical asymptotes, we use test values. For instance, as \( x \to 1^- \), we use a test value close to and to the left of 1 (\( x = 0.9, \) say) to check whether \( y \) is positive or negative to the left of \( x = 1 \):

\[
y = \frac{3(0.9)^2 + 0.9 + 12}{(0.9 - 1)(0.9 - 4)} \quad \text{whose sign is} \quad \frac{(+)}{(-)(-)} \quad \text{(positive)}
\]

So

\[
y \to \infty \quad \text{as} \quad x \to 1^-
\]

On the other hand, as \( x \to 1^+ \), we use a test value close to and to the right of 1 (\( x = 1.1, \) say), to get

\[
y = \frac{3(1.1)^2 + 1.1 + 12}{(1.1 - 1)(1.1 - 4)} \quad \text{whose sign is} \quad \frac{(+)}{(+)(-)} \quad \text{(negative)}
\]

So

\[
y \to -\infty \quad \text{as} \quad x \to 1^+
\]

Similarly, plugging in numbers that are close to 4 from the left and from the right we determine the sign of \( y \) for \( x \)-values near the vertical asymptote \( x = 4 \):

\[
y = \frac{3(3.9)^2 + 3.9 + 12}{(3.9 - 1)(3.9 - 4)} \quad \text{whose sign is} \quad \frac{(+)}{(+)(-)} \quad \text{(negative)}
\]

and

\[
y = \frac{3(4.1)^2 + 4.1 + 12}{(4.1 - 1)(4.1 - 4)} \quad \text{whose sign is} \quad \frac{(+)}{(+)(+)} \quad \text{(positive)}
\]

We conclude that

\[
y \to -\infty \quad \text{as} \quad x \to 4^- \quad \text{and} \quad y \to \infty \quad \text{as} \quad x \to 4^+
\]

**Horizontal Asymptote:** The horizontal asymptote is the value \( y \) approaches as \( x \to \pm \infty \). To help us find this value, we divide both numerator and denominator by \( x^2 \), the highest power of \( x \) that appears in the expression:

\[
y = \frac{3x^2 + x + 12}{x^2 - 5x + 4} = \frac{3x^2 + x + 12}{x^2} \cdot \frac{1}{x^2} = \frac{3x^2 + x + 12}{x^2} \cdot \frac{x^2}{x^2} = \frac{3x^2 + x}{x^2} + \frac{12}{x^2} = 3 + \frac{1}{x} + \frac{12}{x^2}
\]

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One can see that the fractional expressions $\frac{1}{x}$, $\frac{12}{x^2}$, $\frac{5}{x}$, and $\frac{4}{x^2}$ all approach 0 as $x \to \pm \infty$. So as $x \to \pm \infty$, we have

$$y = \frac{3 + \frac{1}{x} + \frac{12}{x^2}}{1 - \frac{5}{x} + \frac{4}{x^2}} \to \frac{3 + 0 + 0}{1 - 0 + 0} = \frac{3}{1} = 3$$

Thus, the horizontal asymptote is the line $y = 3$. Since the graph must approach the horizontal asymptote and

$$y \to \infty \quad \text{as} \quad x \to 1^-$$
$$y \to -\infty \quad \text{as} \quad x \to 1^+$$
$$y \to -\infty \quad \text{as} \quad x \to 4^-$$
$$y \to \infty \quad \text{as} \quad x \to 4^+$$

we can complete it as in the Figure below.

**Domain and Range:** The function $r$ is defined for all values of $x$ other than 1, 4, so the domain is $\{x \mid x \neq 1, 4\}$. From the graph we see that the range is $(-\infty, -13] \cup \left[\frac{11}{9}, \infty\right)$. 
Asymptotes of Rational Functions

Let \( r \) be the rational function
\[
r(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}
\]

1. The vertical asymptotes of \( r \) are the lines \( x = a \), where \( a \) is a zero of the denominator.

2. (a) If \( n < m \), then \( r \) has horizontal asymptote \( y = 0 \).

(b) If \( n = m \), then \( r \) has horizontal asymptote \( y = \frac{a_n}{b_m} \).

(c) If \( n > m \), then \( r \) has no horizontal asymptote.

EXAMPLE: Find the vertical and horizontal asymptotes of \( r(x) = \frac{1}{x} \).

Solution:

**Vertical Asymptotes:** The line \( x = 0 \) is the vertical asymptote because the denominator of \( r \) is zero and the numerator is nonzero when \( x = 0 \).

**Horizontal Asymptote:** The degree of the numerator is less than the degree of the denominator, therefore the horizontal asymptote is the line \( y = 0 \).

EXAMPLE: Find the vertical and horizontal asymptotes of \( r(x) = \frac{1}{x + 3} \).
Example: Find the vertical and horizontal asymptotes of \( r(x) = \frac{1}{x + 3} \).

Solution:

**Vertical Asymptotes:** The line \( x = -3 \) is the vertical asymptote because the denominator of \( r \) is zero and the numerator is nonzero when \( x = -3 \).

**Horizontal Asymptote:** The degree of the numerator is less than the degree of the denominator, therefore the horizontal asymptote is the line \( y = 0 \).

Example: Find the vertical and horizontal asymptotes of \( r(x) = \frac{x}{x - 2} \).

Solution:

**Vertical Asymptotes:** The line \( x = 2 \) is the vertical asymptote because the denominator of \( r \) is zero and the numerator is nonzero when \( x = 2 \).

**Horizontal Asymptote:** The degrees of the numerator and denominator are the same and

\[
\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{1}{1} = 1
\]

Thus, the horizontal asymptote is the line \( y = 1 \).

Example: Find the vertical and horizontal asymptotes of \( r(x) = \frac{5x - 1}{2x + 3} \).
EXAMPLE: Find the vertical and horizontal asymptotes of \( r(x) = \frac{5x - 1}{2x + 3} \).

Solution:

**Vertical Asymptotes**: The line \( x = -3/2 \) is the vertical asymptote because the denominator of \( r \) is zero and the numerator is nonzero when \( x = -3/2 \).

**Horizontal Asymptote**: The degrees of the numerator and denominator are the same and

\[
\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{5}{2}
\]

Thus, the horizontal asymptote is the line \( y = \frac{5}{2} \).

EXAMPLE: Find the vertical and horizontal asymptotes of \( r(x) = \frac{3x^2 - 2x - 1}{2x^2 + 3x - 2} \).

Solution:

**Vertical Asymptotes**: We first factor

\[
r(x) = \frac{3x^2 - 2x - 1}{2x^2 + 3x - 2} = \frac{(3x + 1)(x - 1)}{(2x - 1)(x + 2)}
\]

The lines \( x = \frac{1}{2} \) and \( x = -2 \) are vertical asymptotes because the denominator of \( r \) is zero and the numerator is nonzero when \( x = \frac{1}{2} \) or \( x = -2 \).

**Horizontal Asymptote**: The degrees of the numerator and denominator are the same and

\[
\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{3}{2}
\]

Thus, the horizontal asymptote is the line \( y = \frac{3}{2} \).
EXAMPLE: Find the vertical and horizontal asymptotes of 
\[ r(x) = \frac{x^2 - 4x + 4}{9x^2 - 9x + 2}. \]

Solution:

**Vertical Asymptotes:** We first factor

\[ r(x) = \frac{x^2 - 4x + 4}{9x^2 - 9x + 2} = \frac{(x - 2)^2}{(3x - 1)(3x - 2)} \]

The lines \( x = \frac{1}{3} \) and \( x = \frac{2}{3} \) are vertical asymptotes because the denominator of \( r \) is zero and the numerator is nonzero when \( x = \frac{1}{3} \) or \( x = \frac{2}{3} \).

**Horizontal Asymptote:** The degrees of the numerator and denominator are the same and

\[ \frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{1}{9} \]

Thus, the horizontal asymptote is the line \( y = \frac{1}{9} \).

EXAMPLE: Find the vertical and horizontal asymptotes of 
\[ r(x) = \frac{3x^3 + 6x^2 - 3x - 6}{x^3 - 5x^2 + 6}. \]
EXAMPLE: Find the vertical and horizontal asymptotes of \( r(x) = \frac{3x^3 + 6x^2 - 3x - 6}{x^3 - 5x^2 + 6x} \).

Solution:

**Vertical Asymptotes:** We first factor

\[
r(x) = \frac{3x^3 + 6x^2 - 3x - 6}{x^3 - 5x^2 + 6x} = \frac{3(x^3 + 2x^2 - x - 2)}{x(x^2 - 5x + 6)} = \frac{3[x^2(x + 2) - (x + 2)]}{x(x - 2)(x - 3)}
\]

\[
= \frac{3(x+2)(x^2 - 1)}{x(x-2)(x-3)} = \frac{3(x+2)(x+1)(x-1)}{x(x-2)(x-3)}
\]

The lines \( x = 0, \ x = 2, \) and \( x = 3 \) are vertical asymptotes because the denominator of \( r \) is zero and the numerator is nonzero when \( x = 0, \ x = 2, \) or \( x = 3. \)

**Horizontal Asymptote:** The degrees of the numerator and denominator are the same and

\[
\text{leading coefficient of numerator} \div \text{leading coefficient of denominator} = \frac{3}{1} = 3
\]

Thus, the horizontal asymptote is the line \( y = 3. \)

EXAMPLE: Find the vertical and horizontal asymptotes of \( r(x) = \frac{x^2 - x - 2}{x^2 + 2x - 8}. \)
EXAMPLE: Find the vertical and horizontal asymptotes of \( r(x) = \frac{x^2 - x - 2}{x^2 + 2x - 8} \).

Solution:

**Vertical Asymptote:** We first factor

\[
 r(x) = \frac{x^2 - x - 2}{x^2 + 2x - 8} = \frac{(x - 2)(x + 1)}{(x - 2)(x + 4)}
\]

It follows that

\[
 r(x) = \frac{x + 1}{x + 4} \quad (1)
\]

if \( x \neq 2 \). The line \( x = -4 \) is the vertical asymptote because the denominator of (1) is zero and the numerator is nonzero when \( x = -4 \). The line \( x = 2 \) is *not* the vertical asymptote because the denominator of (1) is nonzero when \( x = 2 \).

**Horizontal Asymptote:** The degrees of the numerator and denominator are the same and

\[
 \frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{1}{1} = 1
\]

Thus, the horizontal asymptote is the line \( y = 1 \).
Graphing Rational Functions

We have seen that asymptotes are important when graphing rational functions. In general, we use the following guidelines to graph rational functions.

**Sketching Graphs of Rational Functions**

1. **Factor.** Factor the numerator and denominator.

2. **Intercepts.** Find the \( x \)-intercepts by determining the zeros of the numerator, and the \( y \)-intercept from the value of the function at \( x = 0 \).

3. **Vertical Asymptotes.** Find the vertical asymptotes by determining the zeros of the denominator, and then see if \( y \rightarrow \infty \) or \( y \rightarrow -\infty \) on each side of each vertical asymptote by using test values.

4. **Horizontal Asymptote.** Find the horizontal asymptote (if any) by dividing both numerator and denominator by the highest power of \( x \) that appears in the denominator, and then letting \( x \rightarrow \pm \infty \).

5. **Sketch the Graph.** Graph the information provided by the first four steps. Then plot as many additional points as needed to fill in the rest of the graph of the function.

**EXAMPLE:** Graph the rational function \( r(x) = \frac{2x^2 + 7x - 4}{x^2 + x - 2} \), and state the domain and range.

**Solution:** We factor the numerator and denominator, find the intercepts and asymptotes, and sketch the graph.

**Factor:** We have

\[
r(x) = \frac{2x^2 + 7x - 4}{x^2 + x - 2} = \frac{(2x - 1)(x + 4)}{(x - 1)(x + 2)}
\]

**\( x \)-Intercepts:** The \( x \)-intercepts are the zeros of the numerator, \( x = \frac{1}{2} \) and \( x = -4 \).

**\( y \)-Intercept:** To find the \( y \)-intercept, we substitute \( x = 0 \) into the original form of the function:

\[
r(0) = \frac{2(0)^2 + 7(0) - 4}{(0)^2 + (0) - 2} = \frac{-4}{-2} = 2
\]

The \( y \)-intercept is 2.

**Vertical Asymptotes:** The lines \( x = 1 \) and \( x = -2 \) are vertical asymptotes because the denominator of \( r \) is zero and the numerator is nonzero when \( x = 1 \) and \( x = -2 \).

**Behavior Near Vertical Asymptotes:** We need to know whether \( y \rightarrow \infty \) or \( y \rightarrow -\infty \) on each side of each vertical asymptote. To determine the sign of \( y \) for \( x \)-values near the vertical asymptotes, we use test values. For instance, as \( x \rightarrow 1^- \), we use a test value close to and to the left of 1 (\( x = 0.9 \), say) to check whether \( y \) is positive or negative to the left of \( x = 1 \):

\[
y = \frac{(2(0.9) - 1)(0.9 + 4)}{(0.9 - 1)(0.9 + 2)} \quad \text{whose sign is } \frac{(+) (+)}{(-)(+)} \quad \text{(negative)}
\]
So \( y \to -\infty \) as \( x \to 1^- \). On the other hand, as \( x \to 1^+ \), we use a test value close to and to the right of 1 \( (x = 1.1, \text{ say}) \), to get

\[
y = \frac{(2(1.1) - 1)(1.1 + 4)}{(1.1 - 1)(1.1 + 2)} \quad \text{whose sign is } \left(\frac{+}{+}\right) \quad \text{positive}
\]

So \( y \to \infty \) as \( x \to 1^+ \). The other entries in the following table are calculated similarly.

<table>
<thead>
<tr>
<th>As ( x \to )</th>
<th>(-2^-)</th>
<th>(-2^+)</th>
<th>(1^-)</th>
<th>(1^+)</th>
</tr>
</thead>
<tbody>
<tr>
<td>the sign of ( y = \frac{(2x - 1)(x + 4)}{(x - 1)(x + 2)} ) is</td>
<td>((-)(+))</td>
<td>((-)(+))</td>
<td>((+)(+))</td>
<td>((+)(+))</td>
</tr>
<tr>
<td>so ( y \to )</td>
<td>(-\infty)</td>
<td>(\infty)</td>
<td>(-\infty)</td>
<td>(\infty)</td>
</tr>
</tbody>
</table>

**Horizontal Asymptote:** The degrees of the numerator and denominator are the same and

\[
\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{2}{1} = 2
\]

Thus, the horizontal asymptote is the line \( y = 2 \).

**Additional Values:**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-6)</td>
<td>0.93</td>
</tr>
<tr>
<td>(-3)</td>
<td>-1.75</td>
</tr>
<tr>
<td>(-1)</td>
<td>4.50</td>
</tr>
<tr>
<td>(1.5)</td>
<td>6.29</td>
</tr>
<tr>
<td>(2)</td>
<td>4.50</td>
</tr>
<tr>
<td>(3)</td>
<td>3.50</td>
</tr>
</tbody>
</table>

**Graph:**

**Domain and Range:** The function \( r \) is defined for all values of \( x \) other than 1, \(-2\), so the domain is

\[
\{x \mid x \neq 1, -2\}
\]

From the graph we see that the range is all real numbers.

**EXAMPLE:** Graph the rational function \( r(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3} \), and state the domain and range.
EXAMPLE: Graph the rational function \( r(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3} \), and state the domain and range.

Solution: We factor the numerator and denominator, find the intercepts and asymptotes, and sketch the graph.

**Factor:** We have

\[
   r(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3} = \frac{(x - 1)(x - 2)}{(x - 1)(x - 3)} \tag{2}
\]

It follows that

\[
   r(x) = \frac{x - 2}{x - 3} \tag{3}
\]

if \( x \neq 1 \).

**x-Intercept:** The \( x \)-intercept is the zero of the numerator of the reduced form (3), \( x = 2 \). Note that \( x = 1 \) is a zero of the numerator of the original form (2), but it is not the \( x \)-intercept, since it is not from the domain of \( r \).

**y-Intercept:** To find the \( y \)-intercept, we substitute \( x = 0 \) into the original form (2):

\[
   r(0) = \frac{(0 - 1)(0 - 2)}{(0 - 1)(0 - 3)} = \frac{(-1)(-2)}{(-1)(-3)} = \frac{2}{3}
\]

The \( y \)-intercept is \( \frac{2}{3} \).

**REMARK:** In this particular example one can find the \( y \)-intercept substituting \( x = 0 \) into the reduced form (3) as well. However, in general we recommend to substitute \( x = 0 \) into the original form to avoid mistakes (see Appendix).

**Vertical Asymptote:** The line \( x = 3 \) is the vertical asymptote because the denominator of (3) is zero and the numerator is nonzero when \( x = 3 \). The line \( x = 1 \) is not the vertical asymptote because the denominator of (3) is nonzero when \( x = 1 \).

**Behavior Near Vertical Asymptote:** We need to know whether \( y \to \infty \) or \( y \to -\infty \) on each side of the vertical asymptote \( x = 3 \). To determine the sign of \( y \) for \( x \)-values near the vertical asymptote, we use test values. For instance, as \( x \to 3^- \), we use a test value close to and to the left of 3 (\( x = 2.9, \) say) to check whether (2) is positive or negative to the left of \( x = 3 \):

\[
   y = \frac{2.9 - 2}{2.9 - 3} = \frac{0.9}{0.1} = 9 \quad \text{whose sign is } (+) \quad \text{(negative)}
\]

So \( y \to -\infty \) as \( x \to 3^- \). On the other hand, as \( x \to 3^+ \), we use a test value close to and to the right of 3 (\( x = 3.1, \) say), to get

\[
   y = \frac{3.1 - 2}{3.1 - 3} = \frac{1.1}{-0.1} = -11 \quad \text{whose sign is } (+) \quad \text{(positive)}
\]

So \( y \to \infty \) as \( x \to 3^+ \).

**Horizontal Asymptote:** The degrees of the numerator and denominator are the same and

\[
   \frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{1}{1} = 1
\]

Thus, the horizontal asymptote is the line \( y = 1 \).

**Domain and Range:** The function \( r \) is defined for all values of \( x \) other than 1, 3, so the domain is \( \{ x \mid x \neq 1, 3 \} \). From the graph we see that the range is \( \{ y \mid y \neq 1/2, 1 \} \).
EXAMPLE: Graph the rational function \( r(x) = \frac{5x + 21}{x^2 + 10x + 25} \), and state the domain and range.

Solution:

**Factor**: \( r(x) = \frac{5x + 21}{(x + 5)^2} \)

**x-Intercepts**: \(-\frac{21}{5}\), from \( 5x + 21 = 0 \)

**y-Intercept**: \( \frac{21}{25} \), because \( r(0) = \frac{5(0) + 21}{(0)^2 + 10(0) + 25} = \frac{21}{25} \)

**Vertical Asymptotes**: \( x = -5 \), from the zeros of the denominator

**Behavior Near Vertical Asymptotes**:

<table>
<thead>
<tr>
<th>As ( x \to )</th>
<th>(-5^-)</th>
<th>(-5^+)</th>
</tr>
</thead>
<tbody>
<tr>
<td>the sign of ( y = \frac{5x + 21}{(x + 5)^2} ) is</td>
<td>(-)</td>
<td>(-)</td>
</tr>
<tr>
<td>so ( y \to )</td>
<td>(-\infty)</td>
<td>(-\infty)</td>
</tr>
</tbody>
</table>

**Horizontal Asymptote**: \( y = 0 \), because degree of numerator is less than degree of denominator

**Additional Values**:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-15)</td>
<td>(-0.5)</td>
</tr>
<tr>
<td>(-10)</td>
<td>(-1.2)</td>
</tr>
<tr>
<td>(-3)</td>
<td>(1.5)</td>
</tr>
<tr>
<td>(-1)</td>
<td>(1.0)</td>
</tr>
<tr>
<td>(3)</td>
<td>(0.6)</td>
</tr>
<tr>
<td>(5)</td>
<td>(0.5)</td>
</tr>
<tr>
<td>(10)</td>
<td>(0.3)</td>
</tr>
</tbody>
</table>

**Graph:**

**Domain and Range**: The domain is \( \{ x \mid x \neq -5 \} \)

From the graph we see that the range is approximately the interval \((-\infty, 1.5]\).
EXAMPLE: Graph the rational function \( r(x) = \frac{x^2 - 3x - 4}{2x^2 + 4x} \), and state the domain and range.

Solution:

**Factor:** \( r(x) = \frac{(x + 1)(x - 4)}{2x(x + 2)} \)

**x-Intercepts:** −1 and 4, from \( x + 1 = 0 \) and \( x - 4 = 0 \)

**y-Intercept:** None, since \( r(0) \) is undefined

**Vertical Asymptotes:** \( x = 0 \) and \( x = -2 \), from the zeros of the denominator

**Behavior Near Vertical Asymptotes:**

<table>
<thead>
<tr>
<th>As ( x \to )</th>
<th>(-2^-)</th>
<th>(-2^+)</th>
<th>(0^-)</th>
<th>(0^+)</th>
</tr>
</thead>
<tbody>
<tr>
<td>the sign of ( y = \frac{(x + 1)(x - 4)}{2x(x + 2)} ) is</td>
<td>((-)(-))</td>
<td>((-)(-))</td>
<td>((+)(-))</td>
<td>((+)(-))</td>
</tr>
<tr>
<td>so ( y \to )</td>
<td>(-\infty)</td>
<td>(-\infty)</td>
<td>(\infty)</td>
<td>(-\infty)</td>
</tr>
</tbody>
</table>

**Horizontal Asymptote:** \( y = \frac{1}{2} \), because degree of numerator is less than degree of denominator and

\[
\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{1}{2}
\]

**Additional Values:**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>−3</td>
<td>2.33</td>
</tr>
<tr>
<td>−2.5</td>
<td>3.90</td>
</tr>
<tr>
<td>−0.5</td>
<td>1.50</td>
</tr>
<tr>
<td>1</td>
<td>−1.00</td>
</tr>
<tr>
<td>3</td>
<td>−0.13</td>
</tr>
<tr>
<td>5</td>
<td>0.09</td>
</tr>
</tbody>
</table>

**Graph:**

**Domain and Range:** The function \( r \) is defined for all values of \( x \) other than 0, −2, so the domain is

\[ \{ x \mid x \neq 0, -2 \} \]

From the graph we see that the range is all real numbers.
Slant Asymptotes and End Behavior

If \( r(x) = \frac{P(x)}{Q(x)} \) is a rational function in which the degree of the numerator is one more than the degree of the denominator, we can use the Division Algorithm to express the function in the form

\[
r(x) = ax + b + \frac{R(x)}{Q(x)}
\]

where the degree of \( R \) is less than the degree of \( Q \) and \( a \neq 0 \). This means that as \( x \to \pm \infty \), \( R(x)/Q(x) \to 0 \), so for large values of \( |x| \), the graph of \( y = r(x) \) approaches the graph of the line \( y = ax + b \). In this situation we say that \( y = ax + b \) is a slant asymptote, or an oblique asymptote.

EXAMPLE: Graph the rational function \( r(x) = \frac{x^2 - 4x - 5}{x - 3} \).

Solution:

Factor: \( r(x) = \frac{(x + 1)(x - 5)}{x - 3} \)

\( x \)-Intercepts: -1 and 5, from \( x + 1 = 0 \) and \( x - 5 = 0 \)

\( y \)-Intercept: \( \frac{5}{3} \), because \( r(0) = \frac{(0)^2 - 4(0) - 5}{0 - 3} = \frac{5}{3} \)

Horizontal Asymptote: None, because degree of numerator is greater than degree of denominator

Vertical Asymptotes: \( x = 3 \), from the zeros of the denominator

Behavior Near Vertical Asymptotes: \( y \to \infty \) as \( x \to 3^- \) and \( y \to -\infty \) as \( x \to 3^+ \)

Slant Asymptote: Since the degree of the numerator is one more than the degree of the denominator, the function has a slant asymptote. Dividing (see the margin), we obtain

\[
r(x) = x - 1 - \frac{8}{x - 3}
\]

Thus, \( y = x - 1 \) is the slant asymptote.

Additional Values:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-1.4</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>-5</td>
</tr>
<tr>
<td>6</td>
<td>2.33</td>
</tr>
</tbody>
</table>

Graph:
EXAMPLE: Find the intercepts and asymptotes of the rational function $r(x) = \frac{x}{x^2 - x}$.

Solution:

**Factor:** We have

$$r(x) = \frac{x}{x^2 - x} = \frac{x}{x(x - 1)}$$

(4)

It follows that

$$r(x) = \frac{1}{x - 1}$$

(5)

if $x \neq 0$.

**x-Intercepts:** There are no $x$-intercepts, since the numerator of the *reduced form* (5) is nonzero. Note that $x = 0$ is the zero of the numerator of the *original form* (4), but it is not the $x$-intercept, since it is not from the domain of $r$.

**y-Intercepts:** There are no $y$-intercepts, since $x = 0$ is not from the domain of $r$. Note that if you plug $x = 0$ into the *reduced form* (5), you get $y = -1$ as the $y$-intercept, which is false.

**Vertical Asymptote:** The line $x = 1$ is the vertical asymptote because the denominator of (5) is zero and the numerator is nonzero when $x = 1$. The line $x = 0$ is not the vertical asymptote because the denominator of (5) is nonzero when $x = 0$.

**Horizontal Asymptote:** Since the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is $y = 0$. 