Section 3.4 Real Zeros of Polynomials

Rational Zeros of Polynomials

To help us understand the next theorem, let's consider the polynomial

$$P(x) = (x-2)(x-3)(x+4) = x^3 - x^2 - 14x + 24$$

From the factored form we see that the zeros of P are 2, 3, and -4. When the polynomial is expanded, the constant 24 is obtained by multiplying $(-2) \times (-3) \times 4$. This means that the zeros of the polynomial are all factors of the constant term. The following generalizes this observation.

Rational Zeros Theorem

If the polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then every rational zero of P is of the form

$$\frac{p}{q}$$

where p is a factor of the constant coefficient a_0
and q is a factor of the leading coefficient a_n.

EXAMPLE: Find the rational zeros of $P(x) = x^3 - 3x + 2$.

Solution: Since the leading coefficient is 1, any rational zero must be a divisor of the constant term 2. So the possible rational zeros are ± 1 and ± 2. We test each of these possibilities.

$$\begin{aligned} P(1) &= (1)^3 - 3(1) + 2 = 0 \\ P(-1) &= (-1)^3 - 3(-1) + 2 = 4 \\ P(2) &= (2)^3 - 3(2) + 2 = 4 \\ P(-2) &= (-2)^3 - 3(-2) + 2 = 0 \end{aligned}$$

The rational zeros of P are 1 and -2.

EXAMPLE: Find the rational zeros of $P(x) = x^3 - 11x^2 + 23x + 35$.

Solution: Since the leading coefficient is 1, any rational zero must be a divisor of the constant term 35. So the possible rational zeros are $\pm 1, \pm 5$, and ± 7. We test each of these possibilities.

$$\begin{aligned} P(1) &= (1)^3 - 11(1)^2 + 23(1) + 35 = 48 \\ P(-1) &= (-1)^3 - 11(-1)^2 + 23(-1) + 35 = 0 \\ P(5) &= (5)^3 - 11(5)^2 + 23(5) + 35 = 0 \\ P(-5) &= (-5)^3 - 11(-5)^2 + 23(-5) + 35 = -480 \\ P(7) &= (7)^3 - 11(7)^2 + 23(7) + 35 = 0 \\ P(-7) &= (-7)^3 - 11(-7)^2 + 23(-7) + 35 = -1008 \end{aligned}$$

The rational zeros of P are $-1, 5$, and 7.

Finding the Rational Zeros of a Polynomial

1. **List Possible Zeros.** List all possible rational zeros using the Rational Zeros Theorem.
2. **Divide.** Use synthetic division to evaluate the polynomial at each of the candidates for rational zeros that you found in Step 1. When the remainder is 0, note the quotient you have obtained.
3. **Repeat.** Repeat Steps 1 and 2 for the quotient. Stop when you reach a quotient that is quadratic or factors easily, and use the quadratic formula or factor to find the remaining zeros.

EXAMPLE: Factor the polynomial $P(x) = 2x^3 + x^2 - 13x + 6$, and find all its zeros.

Solution: By the Rational Zeros Theorem the rational zeros of P are of the form

$$\text{possible rational zero of } P = \frac{\text{factor of constant term}}{\text{factor of leading coefficient}}$$

The constant term is 6 and the leading coefficient is 2, so

$$\text{possible rational zero of } P = \frac{\text{factor of 6}}{\text{factor of 2}}$$

The factors of 6 are $\pm1, \pm2, \pm3, \pm6$ and the factors of 2 are $\pm1, \pm2$. Thus, the possible rational zeros of P are

$$\pm\frac{1}{1}, \quad \pm\frac{2}{1}, \quad \pm\frac{3}{1}, \quad \pm\frac{6}{1}, \quad \pm\frac{1}{2}, \quad \pm\frac{2}{2}, \quad \pm\frac{3}{2}, \quad \pm\frac{6}{2}$$

Simplifying the fractions and eliminating duplicates, we get the following list of possible rational zeros:

$$\pm1, \quad \pm2, \quad \pm3, \quad \pm6, \quad \pm\frac{1}{2}, \quad \pm\frac{3}{2}$$

To check which of these *possible* zeros actually *are* zeros, we need to evaluate P at each of these numbers. An efficient way to do this is to use synthetic division (one can also use long division):

$$\begin{array}{r|rrrr} & 2x^2 & +3x & -10 & \\ \hline x-1 & 2x^3 & +x^2 & -13x & +6 \\ & -2x^3 & +2x^2 & & \\ \hline & & 3x^2 & -13x & \\ & & -3x^2 & +3x & \\ \hline & & & -10x & +6 \\ & & & 10x & -10 \\ \hline & & & & -4 \end{array}$$

Test if 1 is a zero

$$\begin{array}{r|rrrr} 1 & 2 & 1 & -13 & 6 \\ & & 2 & 3 & -10 \\ \hline & 2 & 3 & -10 & -4 \end{array}$$

Remainder is not 0, so 1 is not a zero.

Test if 2 is a zero

$$\begin{array}{r|rrrr} 2 & 2 & 1 & -13 & 6 \\ & & 4 & 10 & -6 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

Remainder is 0, so 2 is a zero.

$$\begin{array}{r|rrrr} & 2x^2 & +5x & -3 & \\ \hline x-2 & 2x^3 & +x^2 & -13x & +6 \\ & -2x^3 & +4x^2 & & \\ \hline & & 5x^2 & -13x & \\ & & -5x^2 & +10x & \\ \hline & & & -3x & +6 \\ & & & 3x & -6 \\ \hline & & & & 0 \end{array}$$

We see that 2 is a zero of P and that P factors as

$$P(x) = 2x^3 + x^2 - 13x + 6 = (x-2)(2x^2 + 5x - 3) = (x-2)(2x-1)(x+3)$$

From the factored form we see that the zeros of P are 2, $\frac{1}{2}$, and -3.

EXAMPLE: Find the zeros of $P(x)=x^4-5x^3-5x^2+23x+10$.

Solution: By the Rational Zeros Theorem the rational zeros of P are of the form

$$\text{possible rational zero of } P \;=\; \frac{\text{factor of constant term}}{\text{factor of leading coefficient}}$$

The constant term is 10 and the leading coefficient is 1, so

$$\text{possible rational zero of } P \;=\; \frac{\text{factor of 10}}{\text{factor of 1}}=\text{factor of 10}$$

The factors of 10 are $\pm1,\pm2,\pm5,\pm10$. Using synthetic (or long) division we find that 1 and 2 are not zeros, but that 5 is a zero:

1	1	−5	−5	23	10
		1	−4	−9	14
	1	−4	−9	14	24

2	1	−5	−5	23	10
		2	−6	−22	2
	1	−3	−11	1	12

5	1	−5	−5	23	10
		5	0	−25	−10
	1	0	−5	−2	0

$$\begin{array}{r|l} & x^3-4x^2-9x+14 \\ x-1 & x^4-5x^3-5x^2+23x+10 \\ & -x^4+x^3 \\ & -4x^3-5x^2 \\ & 4x^3-4x^2 \\ & -9x^2+23x \\ & 9x^2-9x \\ & 14x+10 \\ & -14x+14 \\ & 24 \end{array}$$

$$\begin{array}{r|l} & x^3-3x^2-11x+1 \\ x-2 & x^4-5x^3-5x^2+23x+10 \\ & -x^4+2x^3 \\ & -3x^3-5x^2 \\ & 3x^3-6x^2 \\ & -11x^2+23x \\ & 11x^2-22x \\ & x+10 \\ & -x+2 \\ & 12 \end{array}$$

$$\begin{array}{r|l} & x^3-5x-2 \\ x-5 & x^4-5x^3-5x^2+23x+10 \\ & -x^4+5x^3 \\ & -5x^2+23x \\ & 5x^2-25x \\ & -2x+10 \\ & 2x-10 \\ & 0 \end{array}$$

It follows that P factors as

$$x^4-5x^3-5x^2+23x+10=(x-5)(x^3-5x-2)$$

We now try to factor the quotient x^3-5x-2. Its possible zeros are the divisors of -2, namely,

$$\pm1,\quad \pm2$$

Since we already know that 1 and 2 are not zeros of the original polynomial P, we don't need to try them again. Using synthetic (or long) division we find that -1 is not a zero, but that -2 is a zero:

−2	1	0	−5	−2
		−2	4	2
	1	−2	−1	0

$$\begin{array}{r|l} & x^2-2x-1 \\ x+2 & x^3-5x-2 \\ & -x^3-2x^2 \\ & -2x^2-5x \\ & 2x^2+4x \\ & -x-2 \\ & x+2 \\ & 0 \end{array}$$

$$\begin{array}{r|l} & x^2-x-4 \\ x+1 & x^3-5x-2 \\ & -x^3-x^2 \\ & -x^2-5x \\ & x^2+x \\ & -4x-2 \\ & 4x+4 \\ & 2 \end{array}$$

Therefore P factors as

$$x^4-5x^3-5x^2+23x+10=(x-5)(x^3-5x-2)=(x-5)(x+2)(x^2-2x-1)$$

Now we use the quadratic formula to obtain the two remaining zeros of P:

$$x=\frac{2\pm\sqrt{(-2)^2-4(1)(-1)}}{2}=\left\{\frac{2\pm\sqrt{4+4}}{2}=\frac{2\pm\sqrt{4\cdot2}}{2}=\frac{2\pm\sqrt{4}\sqrt{2}}{2}=\frac{2\pm2\sqrt{2}}{2}\right\}=1\pm\sqrt{2}$$

The zeros of P are $5,\ -2,\ 1+\sqrt{2}$, and $1-\sqrt{2}$.