Section 3.3 Dividing Polynomials

Long Division of Polynomials

Dividing polynomials is much like the familiar process of dividing numbers. When we divide 38 by 7, the quotient is 5 and the remainder is 3. We write

$$\frac{\overset{\text{Dividend}}{38}}{\underset{\text{Divisor}}{7}} = \underset{\text{Quotient}}{5} + \frac{\overset{\text{Remainder}}{3}}{7}$$

To divide polynomials, we use long division, as follows.

> **Division Algorithm**
>
> If $P(x)$ and $D(x)$ are polynomials, with $D(x) \neq 0$, then there exist unique polynomials $Q(x)$ and $R(x)$, where $R(x)$ is either 0 or of degree less than the degree of $D(x)$, such that
>
> $$\underset{\text{Dividend}}{P(x)} = \underset{\text{Divisor}}{D(x)} \cdot \underset{\text{Quotient}}{Q(x)} + \underset{\text{Remainder}}{R(x)}$$
>
> The polynomials $P(x)$ and $D(x)$ are called the **dividend** and **divisor**, respectively, $Q(x)$ is the **quotient**, and $R(x)$ is the **remainder**.

EXAMPLE: Divide $6x^2 - 26x + 12$ by $x - 4$.

Solution: We have

$$\begin{array}{r|l} & 6x \\ \hline x-4 & 6x^2 - 26x + 12 \\ & \underline{6x^2 - 24x} \\ & \quad\;\; -2x + 12 \end{array} \qquad\qquad \begin{array}{r|l} & 6x \;\; - 2 \\ \hline x-4 & 6x^2 - 26x + 12 \\ & \underline{6x^2 - 24x} \\ & \quad\;\; -2x + 12 \\ & \quad\;\; \underline{-2x + \;\;8} \\ & \qquad\qquad\;\; 4 \end{array}$$

The division process ends when the last line is of lesser degree than the divisor. The last line then contains the *remainder*, and the top line contains the *quotient*. The result of the division can be interpreted in either of two ways.

$$\frac{6x^2 - 26x + 12}{x - 4} = 6x - 2 + \frac{4}{x - 4}$$

or

$$\underset{\text{Dividend}}{6x^2 - 26x + 12} = \underset{\text{Divisor}}{(x - 4)}\underset{\text{Quotient}}{(6x - 2)} + \underset{\text{Remainder}}{4}$$

EXAMPLE: Let $P(x) = 8x^4 + 6x^2 - 3x + 1$ and $D(x) = 2x^2 - x + 2$. Find polynomials $Q(x)$ and $R(x)$ such that $P(x) = D(x) \cdot Q(x) + R(x)$.

Solution: We have

$$
\begin{array}{r|l}
 & 4x^2 + 2x \\ \hline
2x^2 - x + 2 & 8x^4 + 0x^3 + 6x^2 - 3x + 1 \\
 & 8x^4 - 4x^3 + 8x^2 \\ \hline
 & \quad 4x^3 - 2x^2 - 3x \\
 & \quad 4x^3 - 2x^2 + 4x \\ \hline
 & \qquad\qquad -7x + 1
\end{array}
\qquad\qquad
\begin{array}{r|l}
 & \qquad\qquad 4x^2 + 2x \\ \hline
2x^2 - x + 2 & 8x^4 \qquad + 6x^2 - 3x + 1 \\
 & -8x^4 + 4x^3 - 8x^2 \\ \hline
 & \quad 4x^3 - 2x^2 - 3x \\
 & \quad -4x^3 + 2x^2 - 4x \\ \hline
 & \qquad\qquad -7x + 1
\end{array}
$$

The process is complete at this point because $-7x + 1$ is of lesser degree than the divisor $2x^2 - x + 2$. From the above long division we see that $Q(x) = 4x^2 + 2x$ and $R(x) = -7x + 1$, so

$$8x^4 + 6x^2 - 3x + 1 = (2x^2 - x + 2)(4x^2 + 2x) + (-7x + 1)$$

EXAMPLE: Divide $x^3 + 2x^2 - 3x + 1$ by $x^2 + 1$.

Solution: We have

$$
\begin{array}{r|l}
 & \qquad\qquad x + 2 \\ \hline
x^2 + 1 & x^3 + 2x^2 - 3x + 1 \\
 & -x^3 \qquad\quad - x \\ \hline
 & \qquad 2x^2 - 4x + 1 \\
 & \qquad -2x^2 \qquad - 2 \\ \hline
 & \qquad\qquad -4x - 1
\end{array}
$$

So $x^3 + 2x^2 - 3x + 1 = (x^2 + 1)(x + 2) + (-4x - 1)$.

EXAMPLE: Divide $2x^3 - 7x^2 + 5$ by $x - 3$.

Solution: We have

Long Division

$$
\begin{array}{r|l}
 & \underbrace{2x^2 - x - 3}_{\text{Quotient}} \\ \hline
x - 3 & 2x^3 - 7x^2 + 0x + 5 \\
 & 2x^3 - 6x^2 \\ \hline
 & \quad -x^2 + 0x \\
 & \quad -x^2 + 3x \\ \hline
 & \qquad -3x + 5 \\
 & \qquad -3x + 9 \\ \hline
 & \qquad\qquad \underbrace{-4}_{\text{Remainder}}
\end{array}
$$

Synthetic Division

$$
\begin{array}{r|rrrr}
3 & 2 & -7 & 0 & 5 \\
 & & 6 & -3 & -9 \\ \hline
 & \multicolumn{3}{l}{\underbrace{2 \quad -1 \quad -3}_{\text{Quotient}}} & \underbrace{-4}_{\text{Remainder}}
\end{array}
$$

So $2x^3 - 7x^2 + 5 = (x - 3)(2x^2 - x - 3) + (-4)$.

The Remainder and Factor Theorems

The next theorem shows how synthetic division can be used to evaluate polynomials easily.

Remainder Theorem

If the polynomial $P(x)$ is divided by $x - c$, then the remainder is the value $P(c)$.

Proof: If the divisor in the Division Algorithm is of the form $x-c$ for some real number c, then the remainder must be a constant (since the degree of the remainder is less than the degree of the divisor). If we call this constant r, then

$$P(x) = (x-c)\cdot Q(x) + r \quad \Longrightarrow \quad P(c) = (c-c)\cdot Q(c) + r = 0 + r = r$$

that is, $P(c)$ is the remainder r.

EXAMPLE: Let $P(x) = 3x^5 + 5x^4 - 4x^3 + 7x + 3$.

(a) Find the quotient and remainder when $P(x)$ is divided by $x+2$.

(b) Use the Remainder Theorem to find $P(-2)$.

Solution:

(a) We have

$$\begin{array}{r|l}
 & 3x^4 - x^3 - 2x^2 + 4x - 1 \\
\hline
x+2 & 3x^5 + 5x^4 - 4x^3 \qquad\quad + 7x + 3 \\
 & -3x^5 - 6x^4 \\
\cline{2-2}
 & \qquad -x^4 - 4x^3 \\
 & \qquad\ \ x^4 + 2x^3 \\
\cline{2-2}
 & \qquad\qquad -2x^3 \\
 & \qquad\qquad 2x^3 + 4x^2 \\
\cline{2-2}
 & \qquad\qquad\qquad 4x^2 + 7x \\
 & \qquad\qquad\quad -4x^2 - 8x \\
\cline{2-2}
 & \qquad\qquad\qquad\qquad -x + 3 \\
 & \qquad\qquad\qquad\qquad\ \ x + 2 \\
\cline{2-2}
 & \qquad\qquad\qquad\qquad\qquad 5
\end{array}
\qquad \text{or} \qquad
\begin{array}{r|rrrrrr}
 & 3 & 5 & -4 & 0 & 7 & 3 \\
-2 & & -6 & 2 & 4 & -8 & 2 \\
\hline
 & 3 & -1 & -2 & 4 & -1 & 5
\end{array}$$

The quotient is $3x^4 - x^3 - 2x^2 + 4x - 1$, and the remainder is 5.

(b) By the Remainder Theorem, $P(-2)$ is the remainder when $P(x)$ is divided by $x-(-2) = x+2$. From part (a) the remainder is 5, so $P(-2) = 5$.

EXAMPLE: Let $P(x) = 8x^5 - 2x^4 + 10x^3 + x^2 - 20x + 10$.

(a) Find the quotient and remainder when $P(x)$ is divided by $x+2$.

(b) Use the Remainder Theorem to find $P(-2)$.

EXAMPLE: Let $P(x) = 8x^5 - 2x^4 + 10x^3 + x^2 - 20x + 10$.

(a) Find the quotient and remainder when $P(x)$ is divided by $x + 2$.

(b) Use the Remainder Theorem to find $P(-2)$.

Solution:

(a) We have

$$
\begin{array}{rrrrrrr}
 & 8x^4 & -18x^3 & +46x^2 & -91x & +162 & \\
\cline{2-7}
x+2) & 8x^5 & -2x^4 & +10x^3 & +x^2 & -20x & +10 \\
 & -8x^5 & -16x^4 & & & & \\
\cline{2-3}
 & & -18x^4 & +10x^3 & & & \\
 & & 18x^4 & +36x^3 & & & \\
\cline{3-4}
 & & & 46x^3 & +x^2 & & \\
 & & & -46x^3 & -92x^2 & & \\
\cline{4-5}
 & & & & -91x^2 & -20x & \\
 & & & & 91x^2 & +182x & \\
\cline{5-6}
 & & & & & 162x & +10 \\
 & & & & & -162x & -324 \\
\cline{6-7}
 & & & & & & -314
\end{array}
$$

or

$$
\begin{array}{r|rrrrrr}
 & 8 & -2 & 10 & 1 & -20 & 10 \\
-2 & & -16 & 36 & -92 & 182 & -324 \\
\hline
 & 8 & -18 & 46 & -91 & 162 & -314
\end{array}
$$

The quotient is $8x^4 - 18x^3 + 46x^2 - 91x + 162$, and the remainder is -314.

(b) By the Remainder Theorem, $P(-2)$ is the remainder when $P(x)$ is divided by $x - (-2) = x + 2$. From part (a) the remainder is 5, so $P(-2) = -314$.

Factor Theorem

c is a zero of P if and only if $x - c$ is a factor of $P(x)$.

EXAMPLE: Let $P(x) = x^3 - 7x + 6$. Show that $P(1) = 0$, and use this fact to factor $P(x)$ completely.

Solution: Substituting, we see that $P(1) = 1^3 - 7 \cdot 1 + 6 = 0$. By the Factor Theorem, this means that $x - 1$ is a factor of $P(x)$. Using long or synthetic division

$$
\begin{array}{rrrr}
 & x^2 & +x & -6 \\
\cline{2-4}
x-1) & x^3 & & -7x+6 \\
 & -x^3 & +x^2 & \\
\cline{2-3}
 & & x^2 & -7x \\
 & & -x^2 & +x \\
\cline{3-4}
 & & & -6x+6 \\
 & & & 6x-6 \\
\cline{4-4}
 & & & 0
\end{array}
\qquad \text{or} \qquad
\begin{array}{r|rrrr}
 & 1 & 0 & -7 & 6 \\
1 & & 1 & 1 & -6 \\
\hline
 & 1 & 1 & -6 & 0
\end{array}
$$

therefore $P(x) = x^3 - 7x + 6 = (x - 1)(x^2 + x - 6) = (x - 1)(x - 2)(x + 3)$.

EXAMPLE: Let $P(x)=x^3+x^2-94x+176$. Factor $P(x)$ completely.

Solution: One can see that $P(2)=2^3+2^2-94\cdot 2+176=0$. By the Factor Theorem, this means that $x-2$ is a factor of $P(x)$. Using long or synthetic division

$$\begin{array}{r}
x^2 \quad +3x \quad -88 \\
x-2\overline{\big)\ x^3 \quad +x^2-94x+176} \\
\underline{-x^3+2x^2} \qquad\qquad\quad \\
3x^2-94x \qquad \\
\underline{-3x^2 \quad +6x} \qquad \\
-88x+176 \\
\underline{88x-176} \\
0
\end{array}
\qquad \text{or} \qquad
\begin{array}{r|rrrr}
 & 1 & 1 & -94 & 176 \\
2 & & 2 & 6 & -176 \\
\hline
 & 1 & 3 & -88 & 0
\end{array}$$

therefore

$$\begin{aligned}
P(x) &= x^3+x^2-94x+176 \\
&= (x-2)(x^2+3x-88) \\
&= (x-2)(x-8)(x+11)
\end{aligned}$$

EXAMPLE: Let $P(x)=x^3+21x^2-157x+135$. Factor $P(x)$ completely.

Solution: One can see that $P(1)=1^3+21\cdot 1^2-157\cdot 1+135=0$. By the Factor Theorem, this means that $x-1$ is a factor of $P(x)$. Using long or synthetic division

$$\begin{array}{r}
x^2 \quad +22x-135 \\
x-1\overline{\big)\ x^3+21x^2-157x+135} \\
\underline{-x^3 \quad +x^2} \qquad\qquad\quad \\
22x^2-157x \qquad \\
\underline{-22x^2 \quad +22x} \qquad \\
-135x+135 \\
\underline{135x-135} \\
0
\end{array}
\qquad \text{or} \qquad
\begin{array}{r|rrrr}
 & 1 & 21 & -157 & 135 \\
1 & & 1 & 22 & -135 \\
\hline
 & 1 & 22 & -135 & 0
\end{array}$$

therefore

$$\begin{aligned}
P(x) &= x^3+21x^2-157x+135 \\
&= (x-1)(x^2+22x-135) \\
&= (x-1)(x-5)(x+27)
\end{aligned}$$

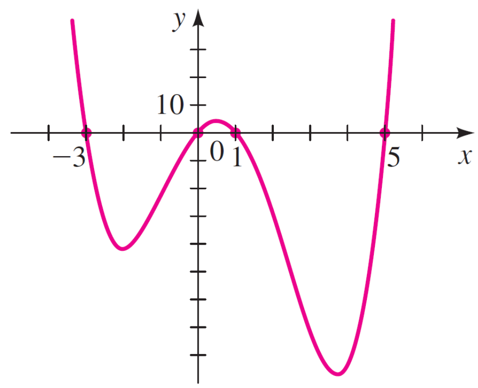
EXAMPLE: Find a polynomial of degree 4 that has zeros $-3, 0, 1$, and 5.

Solution: By the Factor Theorem, $x-(-3),\ x-0,\ x-1$, and $x-5$ must all be factors of the desired polynomial, so let

$$\begin{aligned} P(x) &= (x+3)(x-0)(x-1)(x-5) \\ &= x^4 - 3x^3 - 13x^2 + 15x \end{aligned}$$

Since $P(x)$ is of degree 4 it is a solution of the problem. Any other solution of the problem must be a constant multiple of $P(x)$, since only multiplication by a constant does not change the degree.

The polynomial P is graphed in the Figure below. Note that the zeros of P correspond to the x-intercepts of the graph.



Appendix

EXAMPLE: Divide $x^2 - 2x + 1$ by $x - 1$.

Solution: We have

$$
\begin{array}{r}
x-1 \\
x-1\overline{)\,x^2-2x+1} \\
\underline{-x^2+x} \\
-x+1 \\
\underline{x-1} \\
0
\end{array}
\qquad\text{or}\qquad
\begin{array}{c|ccc}
 & 1 & -2 & 1 \\
1 & & 1 & -1 \\ \hline
 & 1 & -1 & 0
\end{array}
$$

So $x^2 - 2x + 1 = (x-1)(x-1) + 0 = (x-1)^2$.

EXAMPLE: Divide $8x^3 - 3x^2 + 2x - 1$ by $x + 2$.

Solution: We have

$$
\begin{array}{r}
8x^2-19x+40 \\
x+2\overline{)\,8x^3-3x^2+2x-1} \\
\underline{-8x^3-16x^2} \\
-19x^2+2x \\
\underline{19x^2+38x} \\
40x-1 \\
\underline{-40x-80} \\
-81
\end{array}
\qquad\text{or}\qquad
\begin{array}{c|cccc}
 & 8 & -3 & 2 & -1 \\
-2 & & -16 & 38 & -80 \\ \hline
 & 8 & -19 & 40 & -81
\end{array}
$$

So $8x^3 - 3x^2 + 2x - 1 = (x+2)(8x^2 - 19x + 40) + (-81)$.

EXAMPLE: Divide $x^5 + 1$ by $2x^3 + x + 1$.

Solution: We have

$$
\begin{array}{r}
\frac{1}{2}x^2 \phantom{+\frac{1}{4}x} - \frac{1}{4} \\
2x^3+x+1\overline{)\,x^5 \phantom{-\frac{1}{2}x^3-\frac{1}{2}x^2+\frac{1}{4}x} +1} \\
\underline{-x^5-\frac{1}{2}x^3-\frac{1}{2}x^2\phantom{+\frac{1}{4}x+1}} \\
-\frac{1}{2}x^3-\frac{1}{2}x^2\phantom{+\frac{1}{4}x}+1 \\
\underline{\frac{1}{2}x^3\phantom{-\frac{1}{2}x^2}+\frac{1}{4}x+\frac{1}{4}} \\
-\frac{1}{2}x^2+\frac{1}{4}x+\frac{5}{4}
\end{array}
$$

So

$$x^5 + 1 = (2x^3 + x + 1)\left(\frac{1}{2}x^2 - \frac{1}{4}\right) + \left(-\frac{1}{2}x^2 + \frac{1}{4}x + \frac{5}{4}\right)$$