Section 3.3 Dividing Polynomials

Long Division of Polynomials

Dividing polynomials is much like the familiar process of dividing numbers. When we divide 38 by 7, the quotient is 5 and the remainder is 3. We write

\[
\frac{38}{7} = 5 + \frac{3}{7}
\]

To divide polynomials, we use long division, as follows.

**Division Algorithm**

If \( P(x) \) and \( D(x) \) are polynomials, with \( D(x) \neq 0 \), then there exist unique polynomials \( Q(x) \) and \( R(x) \), where \( R(x) \) is either 0 or of degree less than the degree of \( D(x) \), such that

\[ P(x) = D(x) \cdot Q(x) + R(x) \]

The polynomials \( P(x) \) and \( D(x) \) are called the dividend and divisor, respectively, \( Q(x) \) is the quotient, and \( R(x) \) is the remainder.

**EXAMPLE:** Divide \( 6x^2 - 26x + 12 \) by \( x - 4 \).

**Solution:** We have

\[
\begin{array}{c}
6x^2 - 26x + 12 \\
-x \quad -4 \quad \underline{6x^2 - 26x + 12} \\\n\underline{6x^2 - 24x} \\
-2x + 12
\end{array}
\]

The division process ends when the last line is of lesser degree than the divisor. The last line then contains the remainder, and the top line contains the quotient. The result of the division can be interpreted in either of two ways.

\[
\frac{6x^2 - 26x + 12}{x - 4} = 6x - 2 + \frac{4}{x - 4}
\]

or

\[ 6x^2 - 26x + 12 = (x - 4)(6x - 2) + \text{Remainder} \]
EXAMPLE: Let \( P(x) = 8x^4 + 6x^2 - 3x + 1 \) and \( D(x) = 2x^2 - x + 2 \). Find polynomials \( Q(x) \) and \( R(x) \) such that \( P(x) = D(x) \cdot Q(x) + R(x) \).

Solution: We have

\[
\begin{array}{c|ccccc}
& 2x^2 - x + 2 & 2x^2 - x + 2 \\
\hline
2x^2 - x + 2 & 8x^4 + 0x^3 + 6x^2 - 3x + 1 & 8x^4 - 4x^3 + 8x^2 \\
\hline
& 4x^3 - 2x^2 - 3x & 4x^3 - 2x^2 - 3x \\
& 4x^3 - 2x^2 + 4x & -4x^3 + 2x^2 - 4x \\
\hline
& -7x + 1 & -7x + 1
\end{array}
\]

The process is complete at this point because \(-7x + 1\) is of lesser degree than the divisor \(2x^2 - x + 2\). From the above long division we see that \( Q(x) = 4x^2 + 2x \) and \( R(x) = -7x + 1 \), so

\[
8x^4 + 6x^2 - 3x + 1 = (2x^2 - x + 2)(4x^2 + 2x) + (-7x + 1)
\]

EXAMPLE: Divide \( x^3 + 2x^2 - 3x + 1 \) by \( x^2 + 1 \).

Solution: We have

\[
\begin{array}{c|cc}
& x + 2 & x + 2 \\
\hline
x^2 + 1 & x^3 + 2x^2 - 3x + 1 & -x^3 - x \\
\hline
& 2x^2 - 4x + 1 & -2x^2 - 2 \\
& -2x^2 & -4x - 1
\end{array}
\]

So \( x^3 + 2x^2 - 3x + 1 = (x^2 + 1)(x + 2) + (-4x - 1) \).

EXAMPLE: Divide \( 2x^3 - 7x^2 + 5 \) by \( x - 3 \).

Solution: We have

\[
\begin{array}{c|cccc}
3 & 2 & -7 & 0 & 5 \\
\hline
& 6 & -3 & -9 & -4
\end{array}
\]

So \( 2x^3 - 7x^2 + 5 = (x - 3)(2x^2 - x - 3) + (-4) \).
The Remainder and Factor Theorems

The next theorem shows how synthetic division can be used to evaluate polynomials easily.

**Remainder Theorem**

If the polynomial \( P(x) \) is divided by \( x - c \), then the remainder is the value \( P(c) \).

Proof: If the divisor in the Division Algorithm is of the form \( x - c \) for some real number \( c \), then the remainder must be a constant (since the degree of the remainder is less than the degree of the divisor). If we call this constant \( r \), then

\[
P(x) = (x - c) \cdot Q(x) + r \quad \implies \quad P(c) = (c - c) \cdot Q(c) + r = 0 + r = r
\]

that is, \( P(c) \) is the remainder \( r \).

EXAMPLE: Let \( P(x) = 3x^5 + 5x^4 - 4x^3 + 7x + 3 \).

(a) Find the quotient and remainder when \( P(x) \) is divided by \( x + 2 \).

(b) Use the Remainder Theorem to find \( P(-2) \).

Solution:

(a) We have

\[
\begin{array}{cccccc}
& 3x^4 & -x^3 & -2x^2 & +4x & -1 & \\
x+2) & 3x^5 & +5x^4 & -4x^3 & +7x & +3 & \\
& -3x^5 & -6x^4 & & & & \\
& -x^4 & -4x^3 & & & & \\
& x^4 & +2x^3 & & & & \\
& -2x^3 & & & & & \\
& 2x^3 & +4x^2 & & & & \\
& 4x^2 & +7x & & & & \\
& -4x^2 & -8x & & & & \\
& & -x & +3 & & & \\
& & x & +2 & & & \\
& & & & & 5 & \\
\end{array}
\]

The quotient is \( 3x^4 - x^3 - 2x^2 + 4x - 1 \), and the remainder is 5.

(b) By the Remainder Theorem, \( P(-2) \) is the remainder when \( P(x) \) is divided by \( x - (-2) = x + 2 \). From part (a) the remainder is 5, so \( P(-2) = 5 \).

EXAMPLE: Let \( P(x) = 8x^5 - 2x^4 + 10x^3 + x^2 - 20x + 10 \).

(a) Find the quotient and remainder when \( P(x) \) is divided by \( x + 2 \).

(b) Use the Remainder Theorem to find \( P(-2) \).
EXAMPLE: Let \( P(x) = 8x^5 - 2x^4 + 10x^3 + x^2 - 20x + 10 \).
(a) Find the quotient and remainder when \( P(x) \) is divided by \( x + 2 \).
(b) Use the Remainder Theorem to find \( P(-2) \).

Solution:
(a) We have

\[
\begin{array}{c|ccccc}
 x + 2 & 8x^5 & -2x^4 & +10x^3 & +x^2 & -20x & +10 \\
-8x^5 -16x^4 & & & & & & \\
\hline
8x^4 & -18x^3 & +46x^2 & -91x & +162 \\
18x^4 & +36x^3 & & & & \\
-18x^4 & +10x^3 & & & & \\
\hline
46x^3 & +x^2 & & & & \\
-46x^3 & -92x^2 & & & & \\
\hline
-91x^2 & -20x & & & & \\
91x^2 & +182x & & & & \\
\hline
162x & +10 & & & & \\
-162x & -324 & & & & \\
\hline
& -314 & & & & \\
\end{array}
\]

or

\[
\begin{array}{c|ccccc}
 -2 & 8 & -2 & 10 & 1 & -20 & 10 \\
-8 & 16 & 36 & -92 & 182 & -324 \\
\hline
8 & -18 & 46 & -91 & 162 & -314 \\
\end{array}
\]

The quotient is \( 8x^4 - 18x^3 + 46x^2 - 91x + 162 \), and the remainder is \( -314 \).
(b) By the Remainder Theorem, \( P(-2) \) is the remainder when \( P(x) \) is divided by \( x - (-2) = x + 2 \). From part (a) the remainder is \( 5 \), so \( P(-2) = -314 \).

Factor Theorem

\[ c \text{ is a zero of } P \text{ if and only if } x - c \text{ is a factor of } P(x). \]

EXAMPLE: Let \( P(x) = x^3 - 7x + 6 \). Show that \( P(1) = 0 \), and use this fact to factor \( P(x) \) completely.

Solution: Substituting, we see that \( P(1) = 1^3 - 7 \cdot 1 + 6 = 0 \). By the Factor Theorem, this means that \( x - 1 \) is a factor of \( P(x) \). Using long or synthetic division

\[
x - 1 \left| \begin{array}{c}
x^3 & +x & -6 \\
x^3 & +x^2 & & & \\
\hline
x^2 & -7x & & & \\
x^2 & +x & & & \\
\hline
-x^2 & -6x & +6 & & \\
6x & -6 & & & \\
x & & & & 0
\end{array} \right|
\]

or

\[
\begin{array}{c|ccccc}
1 & 1 & 0 & -7 & 6 \\
1 & 1 & 1 & -6 & & \\
\hline
1 & 1 & -6 & 0 & & \\
\end{array}
\]

therefore \( P(x) = x^3 - 7x + 6 = (x - 1)(x^2 + x - 6) = (x - 1)(x - 2)(x + 3) \).
EXAMPLE: Let \( P(x) = x^3 + x^2 - 94x + 176 \). Factor \( P(x) \) completely.

Solution: One can see that \( P(2) = 2^3 + 2^2 - 94 \cdot 2 + 176 = 0 \). By the Factor Theorem, this means that \( x - 2 \) is a factor of \( P(x) \). Using long or synthetic division

\[
\begin{array}{c|ccccc}
  & 1 & 1 & -94 & 176 \\
\hline
2 & 2 & 6 & -176 \\
  & 1 & 3 & -88 & 0 \\
\end{array}
\]

therefore

\[
P(x) = x^3 + x^2 - 94x + 176 = (x - 2)(x^2 + 3x - 88) = (x - 2)(x - 8)(x + 11)
\]

EXAMPLE: Let \( P(x) = x^3 + 21x^2 - 157x + 135 \). Factor \( P(x) \) completely.

Solution: One can see that \( P(1) = 1^3 + 21 \cdot 1^2 - 157 \cdot 1 + 135 = 0 \). By the Factor Theorem, this means that \( x - 1 \) is a factor of \( P(x) \). Using long or synthetic division

\[
\begin{array}{c|cccc}
  & 1 & 21 & -157 & 135 \\
\hline
1 & 1 & 22 & -135 & 0 \\
  & 1 & 22 & -135 & 0 \\
\end{array}
\]

therefore

\[
P(x) = x^3 + 21x^2 - 157x + 135 = (x - 1)(x^2 + 22x - 135) = (x - 1)(x - 5)(x + 27)
\]
EXAMPLE: Find a polynomial of degree 4 that has zeros $-3, 0, 1,$ and $5$.

Solution: By the Factor Theorem, $x - (-3), x - 0, x - 1,$ and $x - 5$ must all be factors of the desired polynomial, so let

$$P(x) = (x + 3)(x - 0)(x - 1)(x - 5)$$

$$= x^4 - 3x^3 - 13x^2 + 15x$$

Since $P(x)$ is of degree 4 it is a solution of the problem. Any other solution of the problem must be a constant multiple of $P(x)$, since only multiplication by a constant does not change the degree.

The polynomial $P$ is graphed in the Figure below. Note that the zeros of $P$ correspond to the $x$-intercepts of the graph.
EXAMPLE: Divide $x^2 - 2x + 1$ by $x - 1$.

Solution: We have

\[
\begin{array}{c|ccc}
& & 1 & -1 & 1 \\
\hline 
x - 1 & 1 & -2 & 1 \\
\hline 
& 1 & -1 & 0 \\
\end{array}
\]

So $x^2 - 2x + 1 = (x - 1)(x - 1) + 0 = (x - 1)^2$.

EXAMPLE: Divide $8x^3 - 3x^2 + 2x - 1$ by $x + 2$.

Solution: We have

\[
\begin{array}{c|cccc}
& & 8 & -3 & 2 & -1 \\
\hline 
x + 2 & 8 & -19 & 40 & & \\
\hline 
& 8 & -16 & 38 & -80 \\
\end{array}
\]

So $8x^3 - 3x^2 + 2x - 1 = (x + 2)(8x^2 - 19x + 40) + (-81)$.

EXAMPLE: Divide $x^5 + 1$ by $2x^3 + x + 1$.

Solution: We have

\[
\begin{array}{c|c}
& \frac{1}{2}x^2 - \frac{1}{4} \\
\hline 
2x^3 + x + 1 & x^5 + 1 \\
\hline 
& - \frac{1}{2}x^3 - \frac{1}{2}x^2 - \frac{1}{4}x - \frac{1}{4} \\
\hline 
& - \frac{1}{2}x^3 - \frac{1}{4}x + \frac{1}{4} \\
\end{array}
\]

So

\[
x^5 + 1 = (2x^3 + x + 1)\left(\frac{1}{2}x^2 - \frac{1}{4}\right) + \left(-\frac{1}{2}x^2 + \frac{1}{4}x + \frac{5}{4}\right)
\]