Section 3.1 Quadratic Functions and Models

DEFINITION: A **quadratic function** is a function f of the form

$$f(x) = ax^2 + bx + c$$

where a, b, and c are real numbers and $a \neq 0$.

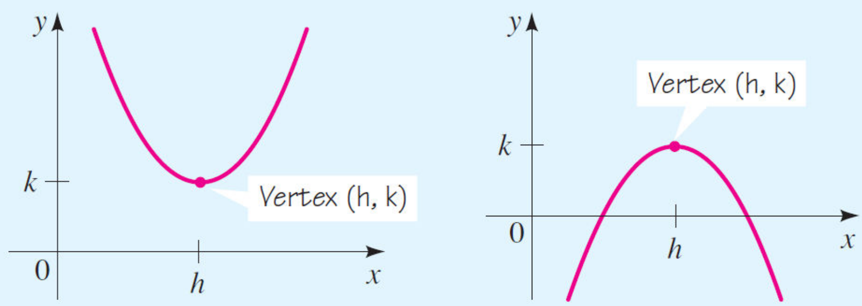
Graphing Quadratic Functions Using the Standard Form

Standard Form of a Quadratic Function

A quadratic function $f(x) = ax^2 + bx + c$ can be expressed in the **standard form**

$$f(x) = a(x - h)^2 + k$$

by completing the square. The graph of f is a parabola with **vertex** (h, k); the parabola opens upward if $a > 0$ or downward if $a < 0$.



$f(x) = a(x - h)^2 + k, \ a > 0$ $\qquad$ $f(x) = a(x - h)^2 + k, \ a < 0$

EXAMPLE: Let $f(x) = x^2 + 10x - 1$. Express f in standard form. Identify the vertex.

Solution: We have

$$\begin{aligned} f(x) &= x^2 + 10x - 1 \\ &= x^2 + 2x \cdot 5 - 1 \\ &= x^2 + 2x \cdot 5 + 5^2 - 5^2 - 1 \\ &= (x + 5)^2 - 26 \\ &= (x - (-5))^2 + (-26) \end{aligned}$$

The vertex is $(-5, -26)$.

EXAMPLE: Let $f(x) = 2x^2 - 12x + 23$.

(a) Express f in standard form.

(b) Sketch the graph of f.

EXAMPLE: Let $f(x) = 2x^2 - 12x + 23$.

(a) Express f in standard form.

(b) Sketch the graph of f.

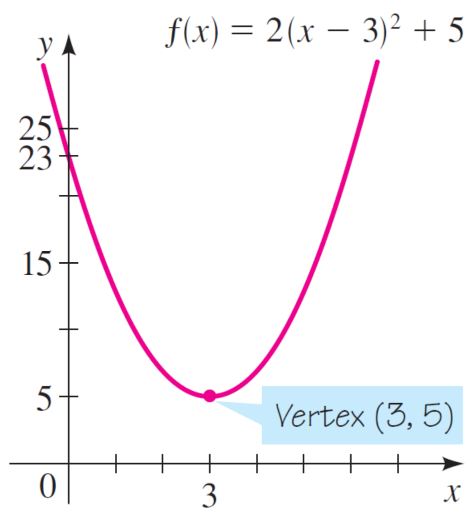
Solution:

(a) We have

$$\begin{aligned} f(x) &= 2x^2 - 12x + 23 \\ &= 2(x^2 - 6x) + 23 \\ &= 2(x^2 - 2x \cdot 3) + 23 \\ &= 2(x^2 - 2x \cdot 3 + 3^2 - 3^2) + 23 \\ &= 2(x^2 - 2x \cdot 3 + 3^2) - 2 \cdot 3^2 + 23 \\ &= 2(x-3)^2 + 5 \end{aligned}$$

The standard form is $f(x) = 2(x-3)^2 + 5$.

(b) The standard form tells us that we get the graph of f by taking the parabola $y = x^2$, shifting it to the right 3 units, stretching it by a factor of 2, and moving it upward 5 units. The vertex of the parabola is at $(3, 5)$, and the parabola opens upward. We sketch the graph in the Figure below after noting that the y-intercept is $f(0) = 23$.



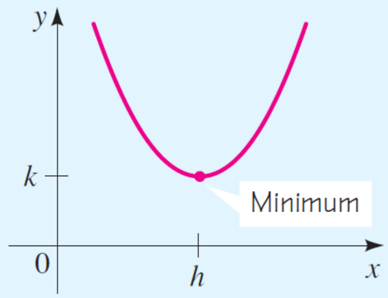
Maximum and Minimum Values of Quadratic Functions

Maximum or Minimum Value of a Quadratic Function

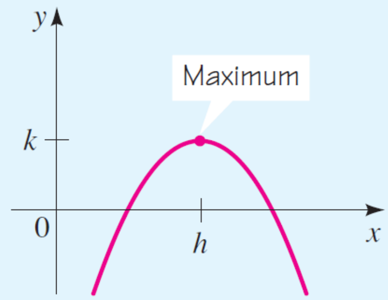
Let f be a quadratic function with standard form $f(x) = a(x-h)^2 + k$. The maximum or minimum value of f occurs at $x = h$.

If $a > 0$, then the **minimum value** of f is $f(h) = k$.

If $a < 0$, then the **maximum value** of f is $f(h) = k$.



$f(x) = a(x-h)^2 + k,\ a > 0$



$f(x) = a(x-h)^2 + k,\ a < 0$

EXAMPLE: Consider the quadratic function $f(x) = 5x^2 - 30x + 49$.

(a) Express f in standard form.

(b) Sketch the graph of f.

(c) Find the minimum value of f.

Solution:

(a) We have

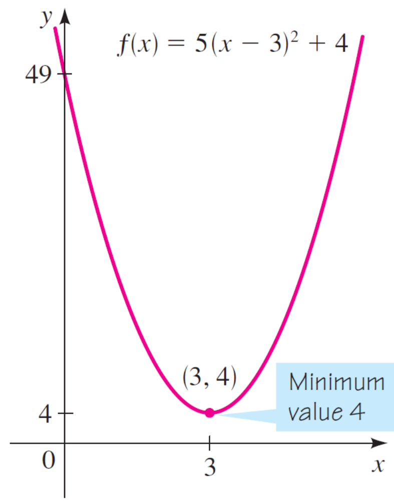
$$\begin{aligned} f(x) &= 5x^2 - 30x + 49 \\ &= 5(x^2 - 6x) + 49 \\ &= 5(x^2 - 2x \cdot 3) + 49 \\ &= 5(x^2 - 2x \cdot 3 + 3^2 - 3^2) + 49 \\ &= 5(x^2 - 2x \cdot 3 + 3^2) - 5 \cdot 3^2 + 49 \\ &= 5(x-3)^2 + 4 \end{aligned}$$

The standard form is $f(x) = 5(x-3)^2 + 4$.

(b)

(c)

(b) The graph is a parabola that has its vertex at $(3, 4)$ and opens upward, as sketched in the Figure below.



(c) Since the coefficient of x^2 is positive, f has a minimum value. The minimum value is $f(3) = 4$.

EXAMPLE: Consider the quadratic function $f(x) = 2x^2 - 8x + 13$.

(a) Express f in standard form.

(b) Sketch the graph of f.

(c) Find the minimum value of f.

EXAMPLE: Consider the quadratic function $f(x) = 2x^2 - 8x + 13$.

(a) Express f in standard form.

(b) Sketch the graph of f.

(c) Find the minimum value of f.

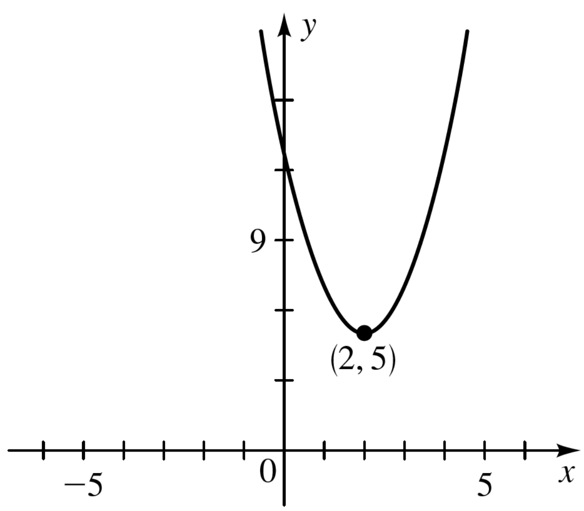
Solution:

(a) We have

$$\begin{aligned} f(x) &= 2x^2 - 8x + 13 \\ &= 2(x^2 - 4x) + 13 \\ &= 2(x^2 - 2x \cdot 2) + 13 \\ &= 2(x^2 - 2x \cdot 2 + 2^2 - 2^2) + 13 \\ &= 2(x^2 - 2x \cdot 2 + 2^2) - 2 \cdot 2^2 + 13 \\ &= 2(x-2)^2 + 5 \end{aligned}$$

The standard form is $f(x) = 2(x-2)^2 + 5$.

(b) The graph is a parabola that has its vertex at $(2, 5)$ and opens upward, as sketched in the Figure below.



(c) Since the coefficient of x^2 is positive, f has a minimum value. The minimum value is $f(2) = 5$.

EXAMPLE: Consider the quadratic function $f(x) = -x^2 + x + 2$.

(a) Express f in standard form.

(b) Sketch the graph of f.

(c) Find the maximum value of f.

EXAMPLE: Consider the quadratic function $f(x) = -x^2 + x + 2$.

(a) Express f in standard form.

(b) Sketch the graph of f.

(c) Find the maximum value of f.

Solution:

(a) We have

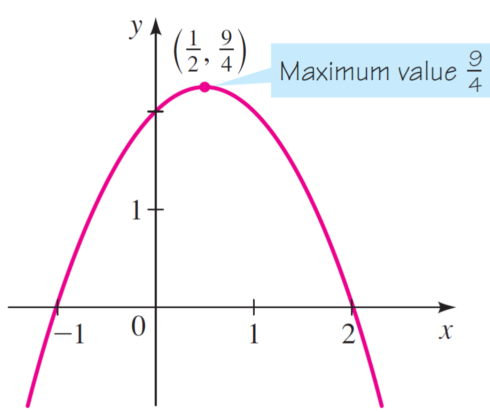
$$\begin{aligned} f(x) &= -x^2 + x + 2 \\ &= -(x^2 - x) + 2 \\ &= -\left(x^2 - 2x \cdot \frac{1}{2}\right) + 2 \\ &= -\left(x^2 - 2x \cdot \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^2}\right) + 2 \\ &= -\left(x^2 - 2x \cdot \frac{1}{2} + \frac{1}{2^2}\right) + \frac{1}{2^2} + 2 \\ &= -\left(x - \frac{1}{2}\right)^2 + \frac{9}{4} \end{aligned}$$

The standard form is $f(x) = -\left(x - \frac{1}{2}\right)^2 + \frac{9}{4}$.

(b) From the standard form we see that the graph is a parabola that opens downward and has vertex $\left(\frac{1}{2}, \frac{9}{4}\right)$. As an aid to sketching the graph, we find the intercepts. The y-intercept is $f(0) = 2$. To find the x-intercepts, we set $f(x) = 0$ and factor the resulting equation.

$$\begin{aligned} -x^2 + x + 2 &= 0 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \end{aligned}$$

Thus, the x-intercepts are $x = 2$ and $x = -1$. The graph of f is sketched in the Figure below.



(c) Since the coefficient of x^2 is negative, f has a maximum value, which is $f\left(\frac{1}{2}\right) = \frac{9}{4}$.

Expressing a quadratic function in standard form helps us sketch its graph as well as find its maximum or minimum value. If we are interested only in finding the maximum or minimum value, then a formula is available for doing so. This formula is obtained by completing the square for the general quadratic function as follows:

$$\begin{aligned}
f(x) &= ax^2 + bx + c \\
&= ax^2 + a \cdot \frac{b}{a}x + c \\
&= a\left(x^2 + \frac{b}{a}x\right) + c \\
&= a\left(x^2 + 2x\frac{b}{2a}\right) + c \\
&= a\left(x^2 + 2x\frac{b}{2a} + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c \\
&= a\left(x^2 + 2x\frac{b}{2a} + \left(\frac{b}{2a}\right)^2\right) - a\left(\frac{b}{2a}\right)^2 + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - a\frac{b^2}{4a^2} + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c
\end{aligned}$$

Maximum or Minimum Value of a Quadratic Function

The maximum or minimum value of a quadratic function $f(x) = ax^2 + bx + c$ occurs at

$$x = -\frac{b}{2a}$$

If $a > 0$, then the **minimum value** is $f\left(-\frac{b}{2a}\right)$.

If $a < 0$, then the **maximum value** is $f\left(-\frac{b}{2a}\right)$.

EXAMPLE: Find the maximum or minimum value of each quadratic function.

(a) $f(x) = x^2 + 4x$ (b) $g(x) = -2x^2 + 4x - 5$

EXAMPLE: Find the maximum or minimum value of each quadratic function.

(a) $f(x) = x^2 + 4x$ (b) $g(x) = -2x^2 + 4x - 5$

Solution:

(a) This is a quadratic function with $a = 1$ and $b = 4$. Thus, the maximum or minimum value occurs at

$$x = -\frac{b}{2a} = -\frac{4}{2 \cdot 1} = -2$$

Since $a > 0$, the function has the *minimum* value

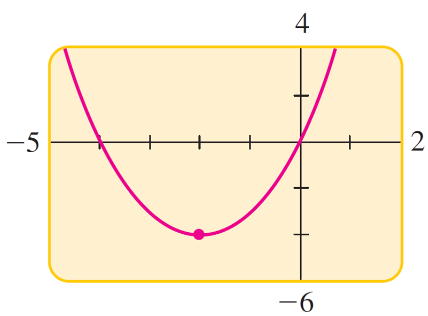
$$f(-2) = (-2)^2 + 4(-2) = -4$$

(b) This is a quadratic function with $a = -2$ and $b = 4$. Thus, the maximum or minimum value occurs at

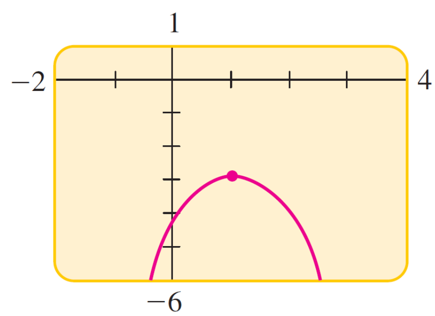
$$x = -\frac{b}{2a} = -\frac{4}{2 \cdot (-2)} = 1$$

Since $a < 0$, the function has the *maximum* value

$$f(1) = -2(1)^2 + 4(1) - 5 = -3$$



The minimum value occurs at $x = -2$.



The maximum value occurs at $x = 1$.

Modeling with Quadratic Functions

EXAMPLE: Most cars get their best gas mileage when traveling at a relatively modest speed. The gas mileage M for a certain new car is modeled by the function

$$M(s) = -\frac{1}{28}s^2 + 3s - 31, \quad 15 \leq s \leq 70$$

where s is the speed in mi/h and M is measured in mi/gal. What is the car's best gas mileage, and at what speed is it attained?

EXAMPLE: Most cars get their best gas mileage when traveling at a relatively modest speed. The gas mileage M for a certain new car is modeled by the function

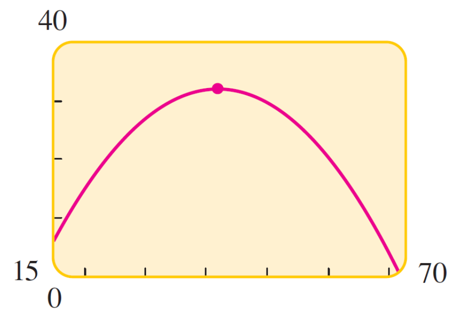
$$M(s) = -\frac{1}{28}s^2 + 3s - 31, \quad 15 \leq s \leq 70$$

where s is the speed in mi/h and M is measured in mi/gal. What is the car's best gas mileage, and at what speed is it attained?

Solution: The function M is a quadratic function with $a = -\frac{1}{28}$ and $b = 3$. Thus, its maximum value occurs when

$$s = -\frac{b}{2a} = -\frac{3}{2\left(-\frac{1}{28}\right)} = 42$$

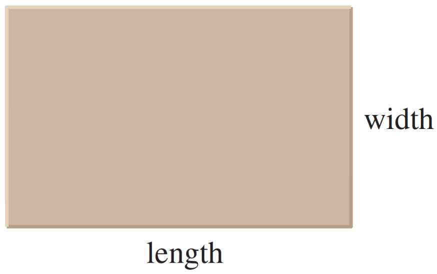
The maximum is $M(42) = -\frac{1}{28}(42)^2 + 3(42) - 31 = 32$. So the car's best gas mileage is 32 mi/gal, when it is traveling at 42 mi/h.

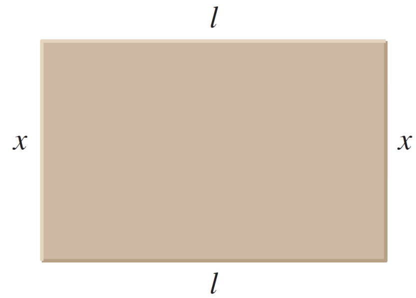


The maximum gas mileage occurs at 42 mi/h.

EXAMPLE: A gardener has 140 feet of fencing to fence in a rectangular vegetable garden. Find the dimensions of the largest area she can fence.

EXAMPLE: A gardener has 140 feet of fencing to fence in a rectangular vegetable garden. Find the dimensions of the largest area she can fence.





Solution: We are asked to find the dimensions of the largest area the gardener can fence. So we let

$$x = \text{ the width of garden}$$

Then we translate the information in the Figure above into the language of algebra:

In Words	In Algebra
Width	x
Length	$70 - x$

The model is the function A that gives the area of the garden for any width x.

$$\boxed{\text{area}} = \boxed{\text{width}} \times \boxed{\text{length}}$$

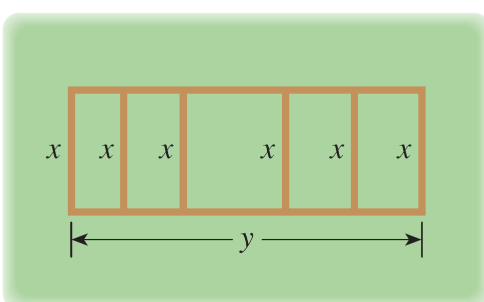
$$A(x) = x(70 - x) = 70x - x^2$$

The area she can fence is modeled by the function $A(x) = 70x - x^2$. We need to find the maximum value of this function. Since this is a quadratic function with $a = -1$ and $b = 70$, the maximum occurs at

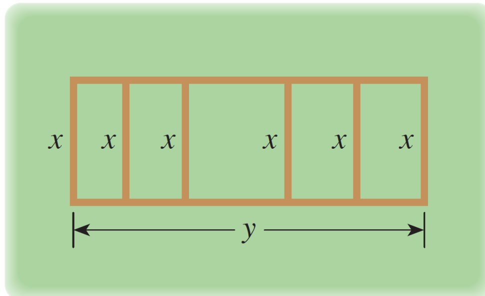
$$x = -\frac{b}{2a} = -\frac{70}{2(-1)} = 35$$

So the maximum area that she can fence has width 35 ft and length $70 - 35 = 35$ ft.

EXAMPLE: If 1800 ft of fencing is available to build five adjacent pens, as shown in the diagram below, express the total area of the pens as a function of x. What value of x will maximize the total area? What is the maximum area?



EXAMPLE: If 1800 ft of fencing is available to build five adjacent pens, as shown in the diagram below, express the total area of the pens as a function of x. What value of x will maximize the total area? What is the maximum area?



Solution: Let

$$x = \text{ width} \quad \text{and} \quad y = \text{ length}$$

Since

$$6x + 2y = 1800$$

it follows that

$$2y = 1800 - 6x \quad \Longrightarrow \quad y = 900 - 3x$$

The model is the function A that gives the area for any width x.

$$\boxed{\text{area}} = \boxed{\text{width}} \times \boxed{\text{length}}$$

$$A(x) = x(900 - 3x) = 900x - 3x^2$$

It follows that the area will attain its maximum when

$$x = -\frac{b}{2a} = -\frac{900}{2(-3)} = 150 \text{ ft}$$

Using this we can find that the maximum area will be

$$A(x) = -3 \cdot 150^2 + 900 \cdot 150 = 67,500 \text{ ft}$$

EXAMPLE: A hockey team plays in an arena that has a seating capacity of 15,000 spectators. With the ticket price set at \$14, average attendance at recent games has been 9500. A market survey indicates that for each dollar the ticket price is lowered, the average attendance increases by 1000.

(a) Find a function that models the revenue in terms of ticket price.

(b) What ticket price is so high that no one attends, and hence no revenue is generated?

(c) Find the price that maximizes revenue from ticket sales.

EXAMPLE: A hockey team plays in an arena that has a seating capacity of 15,000 spectators. With the ticket price set at \$14, average attendance at recent games has been 9500. A market survey indicates that for each dollar the ticket price is lowered, the average attendance increases by 1000.

(a) Find a function that models the revenue in terms of ticket price.

(b) What ticket price is so high that no one attends, and hence no revenue is generated?

(c) Find the price that maximizes revenue from ticket sales.

Solution:

(a) The model we want is a function that gives the revenue for any ticket price. We know that

$$\boxed{\text{revenue}} = \boxed{\text{ticket price}} \times \boxed{\text{attendance}}$$

There are two varying quantities: ticket price and attendance. Since the function we want depends on price, we let

$$x = \text{ ticket price}$$

Next, we must express the attendance in terms of x.

In Words	In Algebra
Ticket price	x
Amount ticket price is lowered	$14 - x$
Increase in attendance	$1000(14 - x)$
Attendance	$9500 + 1000(14 - x) = 23{,}500 - 1000x$

The model is the function R that gives the revenue for a given ticket price x.

$$\boxed{\text{revenue}} = \boxed{\text{ticket price}} \times \boxed{\text{attendance}}$$

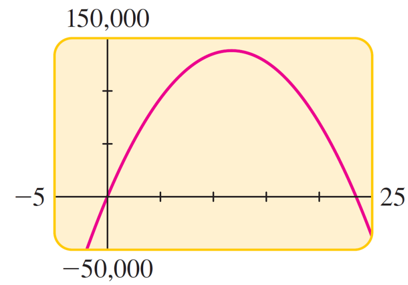
$$R(x) = x(23,500 - 1000x)$$

$$R(x) = 23,500x - 1000x^2$$

We use the model to answer the questions in parts (b) and (c).

(b) We want to find the ticket price x for which

$$R(x) = 23,500x - 1000x^2 = 0$$



We can solve this quadratic equation algebraically or graphically. From the graph in the Figure we see that $R(x) = 0$ when $x = 0$ or $x = 23.5$. So, according to our model, the revenue would drop to zero if the ticket price is \$23.50 or higher. (Of course, revenue is also zero if the ticket price is zero!)

(c) Since $R(x) = 23,500x - 1000x^2$ is a quadratic function with $a = -1000$ and $b = 23,500$, the maximum occurs at

$$x = -\frac{b}{2a} = -\frac{23,500}{2(-1000)} = 11.75$$

So a ticket price of \$11.75 yields the maximum revenue. At this price the revenue is

$$R(11.75) = 23,500(11.75) - 1000(11.75)^2 = \$138,062.50$$