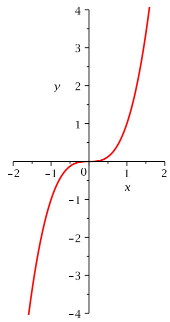
Section 2.7 One-to-One Functions and Their Inverses

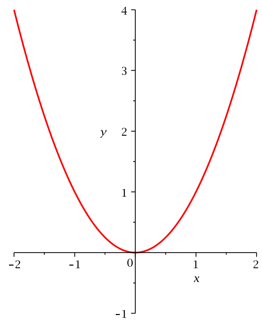
One-to-One Functions

Definition of a One-to-one Function

A function with domain A is called a **one-to-one function** if no two elements of A have the same image, that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$





HORIZONTAL LINE TEST: A function is one-to-one if and only if no horizontal line intersects its graph more than once.

EXAMPLES:

1. Functions $x,\ x^3,\ x^5,\ 1/x$, etc. are one-to-one, since if $x_1 \neq x_2$, then

$$x_1 \neq x_2, \quad x_1^3 \neq x_2^3, \quad x_1^5 \neq x_2^5, \quad \frac{1}{x_1} \neq \frac{1}{x_2}$$

2. The function $f(x) = 4x - 3$ is one-to-one. In fact, suppose x_1 and x_2 are real numbers such that $f(x_1) = f(x_2)$. Then

$$4x_1 - 3 = 4x_2 - 3 \quad \Longleftrightarrow \quad 4x_1 = 4x_2 \quad \Longleftrightarrow \quad x_1 = x_2$$

Therefore, f is one-to-one.

3. The function $f(x) = \dfrac{\sqrt[5]{x}}{2} + 1$ is one-to-one. In fact, suppose x_1 and x_2 are real numbers such that $f(x_1) = f(x_2)$. Then

$$\frac{\sqrt[5]{x_1}}{2} + 1 = \frac{\sqrt[5]{x_2}}{2} + 1$$

$$\frac{\sqrt[5]{x_1}}{2} = \frac{\sqrt[5]{x_2}}{2}$$

$$\sqrt[5]{x_1} = \sqrt[5]{x_2}$$

$$x_1 = x_2$$

Therefore, f is one-to-one.

4. Functions $x^2,\ x^4,\ \sin x$, etc. are not one-to-one, since

$$(-1)^2 = 1^2, \quad (-1)^4 = 1^4, \quad \sin 0 = \sin \pi$$

DEFINITION: Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y \qquad (*)$$

for any y in B.

So, we can reformulate $(*)$ as

$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$ for every x in the domain of f
$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}

IMPORTANT: Do not confuse f^{-1} with $\dfrac{1}{f}$.

EXAMPLES:

1. Let $f(x) = x^3$, then $f^{-1}(x) = \sqrt[3]{x}$, since

$$f^{-1}(f(x)) = f^{-1}(x^3) = \sqrt[3]{x^3} = x \quad \text{and} \quad f(f^{-1}(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$$

2. Let $f(x) = x^3 + 1$, then $f^{-1}(x) = \sqrt[3]{x-1}$, since

$$f^{-1}(f(x)) = f^{-1}(x^3+1) = \sqrt[3]{(x^3+1)-1} = x \quad \text{and} \quad f(f^{-1}(x)) = f(\sqrt[3]{x-1}) = (\sqrt[3]{x-1})^3+1 = x$$

3. Let $f(x) = 2x$, then $f^{-1}(x) = \dfrac{1}{2}x$, since

$$f^{-1}(f(x)) = f^{-1}(2x) = \frac{1}{2}(2x) = x \quad \text{and} \quad f(f^{-1}(x)) = f\left(\frac{1}{2}x\right) = 2\left(\frac{1}{2}x\right) = x$$

4. Let $f(x) = x$, then $f^{-1}(x) = x$, since

$$f^{-1}(f(x)) = f^{-1}(x) = x \quad \text{and} \quad f(f^{-1}(x)) = f(x) = x$$

5. Let $f(x) = 7x + 2$, then $f^{-1}(x) = \dfrac{x-2}{7}$, since

$$f^{-1}(f(x)) = f^{-1}(7x+2) = \frac{(7x+2)-2}{7} = x \quad \text{and} \quad f(f^{-1}(x)) = f\left(\frac{x-2}{7}\right) = 7\left(\frac{x-2}{7}\right)+2 = x$$

Solution:

Step 1: Replace $f(x)$ by y:

$$y = 7x + 2$$

Step 2: Solve for x:

$$y = 7x + 2 \implies y - 2 = 7x \implies \frac{y-2}{7} = x$$

Step 3: Replace x by $f^{-1}(x)$ and y by x:

$$f^{-1}(x) = \frac{x-2}{7}$$

6. Let $f(x) = (3x-2)^5 + 2$. Find $f^{-1}(x)$.

6. Let $f(x) = (3x-2)^5 + 2$. Find $f^{-1}(x)$.

Solution:

Step 1: Replace $f(x)$ by y:

$$y = (3x-2)^5 + 2$$

Step 2: Solve for x:

$$y = (3x-2)^5+2 \quad \Longrightarrow \quad y-2 = (3x-2)^5 \quad \Longrightarrow \quad \sqrt[5]{y-2} = 3x-2 \quad \Longrightarrow \quad \sqrt[5]{y-2}+2 = 3x$$

therefore

$$x = \frac{\sqrt[5]{y-2}+2}{3}$$

Step 3: Replace x by $f^{-1}(x)$ and y by x:

$$f^{-1}(x) = \frac{\sqrt[5]{x-2}+2}{3}$$

7. Let $f(x) = \dfrac{3x-5}{4-2x}$. Find $f^{-1}(x)$.

8. Let $f(x) = \sqrt{x}$. Find $f^{-1}(x)$.

7. Let $f(x) = \dfrac{3x-5}{4-2x}$, then $f^{-1}(x) = \dfrac{4x+5}{3+2x}$.

Solution:

Step 1: Replace $f(x)$ by y:

$$y = \frac{3x-5}{4-2x}$$

Step 2: Solve for x:

$$y = \frac{3x-5}{4-2x} \implies y(4-2x) = 3x-5 \implies 4y - 2xy = 3x - 5 \implies 4y + 5 = 3x + 2xy$$

therefore

$$4y + 5 = x(3+2y) \implies \frac{4y+5}{3+2y} = x$$

Step 3: Replace x by $f^{-1}(x)$ and y by x:

$$f^{-1}(x) = \frac{4x+5}{3+2x}$$

8. Let $f(x) = \sqrt{x}$, then $f^{-1}(x) = x^2,\ x \geq 0$.

IMPORTANT:

domain of $f^{-1} =$ range of f
range of $f^{-1} =$ domain of f

9. Let $f(x) = \sqrt{3-x}$, then $f^{-1}(x) = 3 - x^2,\ x \geq 0$.

Solution:

Step 1: Replace $f(x)$ by y:

$$y = \sqrt{3-x}$$

Step 2: Solve for x:

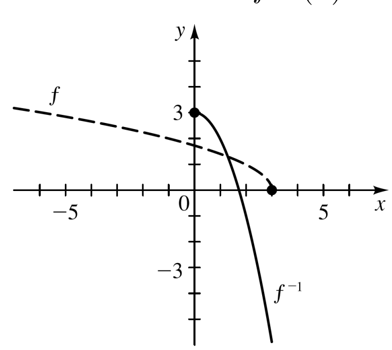
$$y = \sqrt{3-x} \implies y^2 = 3 - x \implies x = 3 - y^2$$

Step 3: Replace x by $f^{-1}(x)$ and y by x:

$$f^{-1}(x) = 3 - x^2$$

Since the range of $f(x)$ is all nonnegative numbers, it follows that the domain of $f^{-1}(x)$ is $x \geq 0$. So,

$$f^{-1}(x) = 3 - x^2,\ x \geq 0$$



10. Let $f(x)=\sqrt{3x-2}$, then $f^{-1}(x)=\dfrac{1}{3}(x^2+2),\ x\geq 0$ (see Appendix, page 7).

11. Let $f(x)=\sqrt[4]{x-1}$, then $f^{-1}(x)=x^4+1,\ x\geq 0$ (see Appendix, page 7).

12. Let $f(x)=\sqrt{x+5}+1$, then $f^{-1}(x)=(x-1)^2-5,\ x\geq 1$ (see Appendix, page 8).

13. Let $f(x)=\sqrt[4]{2x-7}+5$, then $f^{-1}(x)=\dfrac{(x-5)^4+7}{2},\ x\geq 5$ (see Appendix, page 8).

14. The function $f(x)=x^2$ is not invertible, since it is not a one-to-one function.

REMARK: Similarly,

$$x^4, \quad x^{10}, \quad \sin x, \quad \cos x, \quad \text{etc.}$$

are not invertable functions.

15. The function $f(x)=(x+1)^2$ is not invertible.

16. Let $f(x)=x^2, x\geq 0$, then $f^{-1}(x)=\sqrt{x}, x\geq 0$.

17. Let $f(x)=x^2, x\geq 2$, then $f^{-1}(x)=\sqrt{x}, x\geq 4$.

18. Let $f(x)=x^2, x<-3$, then $f^{-1}(x)=-\sqrt{x}, x>9$.

19. The function $f(x)=x^2, x>-1$ is not invertible.

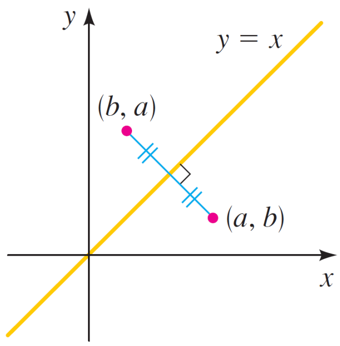
20. Let $f(x)=(x+1)^2, x>3$. Find $f^{-1}(x)$.

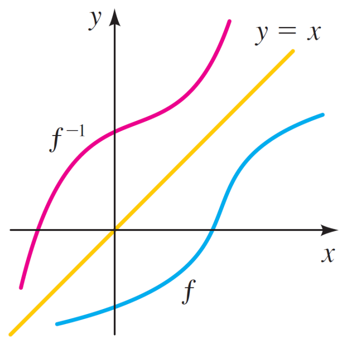
21. Let $f(x)=(1+2x)^2, x\leq -1$. Find $f^{-1}(x)$.

20. Let $f(x)=(x+1)^2, x>3$, then $f^{-1}(x)=\sqrt{x}-1, x>16$ (see Appendix, page 9).

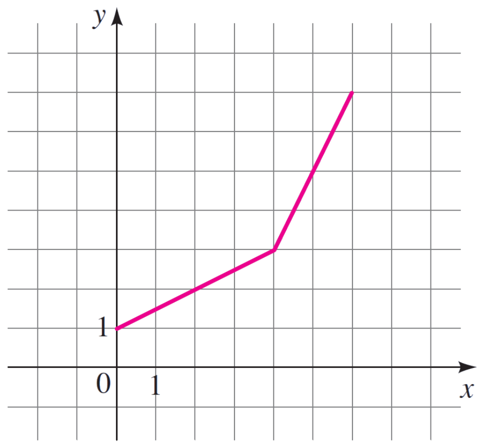
21. Let $f(x)=(1+2x)^2, x\leq -1$, then $f^{-1}(x)=-\dfrac{\sqrt{x}+1}{2}, x\geq 1$ (see Appendix, page 9).

THEOREM: If f has an inverse function f^{-1}, then the graphs of $y=f(x)$ and $y=f^{-1}(x)$ are reflections of one another about the line $y=x$; that is, each is the mirror image of the other with respect to that line.

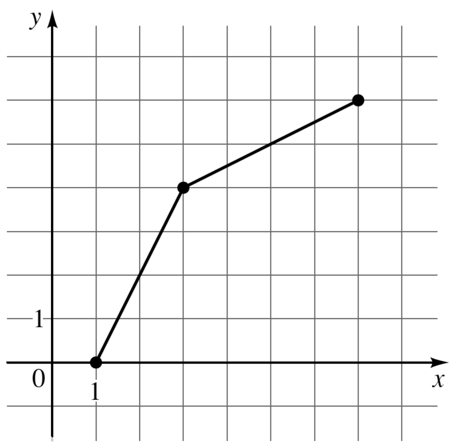




EXAMPLE: The graph of a function f is given. Sketch the graph of f^{-1}.



Solution:



Appendix

10. Let $f(x) = \sqrt{3x-2}$, then $f^{-1}(x) = \dfrac{1}{3}(x^2+2),\ x \geq 0.$

Solution:

Step 1: Replace $f(x)$ by y:

$$y = \sqrt{3x-2}$$

Step 2: Solve for x:

$$y = \sqrt{3x-2} \quad \Longrightarrow \quad y^2 = 3x-2 \quad \Longrightarrow \quad y^2+2 = 3x$$

therefore

$$x = \frac{1}{3}(y^2+2)$$

Step 3: Replace x by $f^{-1}(x)$ and y by x:

$$f^{-1}(x) = \frac{1}{3}(x^2+2)$$

Finally, since the range of f is all nonnegative numbers, it follows that the domain of f^{-1} is $x \geq 0$.

11. Let $f(x) = \sqrt[4]{x-1}$, then $f^{-1}(x) = x^4+1,\ x \geq 0.$

Solution:

Step 1: Replace $f(x)$ by y:

$$y = \sqrt[4]{x-1}$$

Step 2: Solve for x:

$$y = \sqrt[4]{x-1} \quad \Longrightarrow \quad y^4 = x-1$$

therefore

$$x = y^4+1$$

Step 3: Replace x by $f^{-1}(x)$ and y by x:

$$f^{-1}(x) = x^4+1$$

Finally, since the range of f is all nonnegative numbers, it follows that the domain of f^{-1} is $x \geq 0$.

12. Let $f(x)=\sqrt{x+5}+1$, then $f^{-1}(x)=(x-1)^2-5,\ x\geq 1$.

Solution:

Step 1: Replace $f(x)$ by y:

$$y=\sqrt{x+5}+1$$

Step 2: Solve for x:

$$y=\sqrt{x+5}+1 \quad\Longrightarrow\quad y-1=\sqrt{x+5} \quad\Longrightarrow\quad (y-1)^2=x+5$$

therefore

$$x=(y-1)^2-5$$

Step 3: Replace x by $f^{-1}(x)$ and y by x:

$$f^{-1}(x)=(x-1)^2-5$$

Finally, since the range of f is all numbers ≥ 1, it follows that the domain of f^{-1} is $x\geq 1$.

13. Let $f(x)=\sqrt[4]{2x-7}+5$, then $f^{-1}(x)=\dfrac{(x-5)^4+7}{2},\ x\geq 5$.

Solution:

Step 1: Replace $f(x)$ by y:

$$y=\sqrt[4]{2x-7}+5$$

Step 2: Solve for x:

$$y=\sqrt[4]{2x-7}+5 \quad\Longrightarrow\quad y-5=\sqrt[4]{2x-7} \quad\Longrightarrow\quad (y-5)^4=2x-7 \quad\Longrightarrow\quad (y-5)^4+7=2x$$

therefore

$$x=\frac{(y-5)^4+7}{2}$$

Step 3: Replace x by $f^{-1}(x)$ and y by x:

$$f^{-1}(x)=\frac{(x-5)^4+7}{2}$$

Finally, since the range of f is all numbers ≥ 5, it follows that the domain of f^{-1} is $x\geq 5$.

20. Let $f(x) = (x+1)^2, x > 3$, then $f^{-1}(x) = \sqrt{x} - 1, x > 16$.

Solution:

Step 1: Replace $f(x)$ by y:

$$y = (x+1)^2$$

Step 2: Solve for x:

$$y = (x+1)^2 \quad \Longrightarrow \quad \pm\sqrt{y} = x + 1$$

Since x is positive, it follows that $\sqrt{y} = x + 1$, therefore

$$x = \sqrt{y} - 1$$

Step 3: Replace x by $f^{-1}(x)$ and y by x:

$$f^{-1}(x) = \sqrt{x} - 1$$

To find the domain of f^{-1} we note that the range of f is all numbers > 16. Indeed, since $x > 3$, we have

$$f(x) = (x+1)^2 > (3+1)^2 = 4^2 = 16$$

From this it follows that the domain of f^{-1} is $x > 16$.

21. Let $f(x) = (1+2x)^2, x \leq -1$, then $f^{-1}(x) = -\frac{\sqrt{x}+1}{2}, x \geq 1$.

Solution:

Step 1: Replace $f(x)$ by y:

$$y = (1+2x)^2$$

Step 2: Solve for x:

$$y = (1+2x)^2 \quad \Longrightarrow \quad \pm\sqrt{y} = 1 + 2x$$

Since $x \leq -1$, it follows that $-\sqrt{y} = 1 + 2x$, hence

$$-\sqrt{y} - 1 = 2x \quad \Longrightarrow \quad -\frac{\sqrt{y}+1}{2} = x$$

Step 3: Replace x by $f^{-1}(x)$ and y by x:

$$f^{-1}(x) = -\frac{\sqrt{x}+1}{2}$$

To find the domain of f^{-1} we note that the range of f is all numbers ≥ 1. Indeed, since $x \leq -1$, we have

$$f(x) = (1+2x)^2 \geq (1 + 2 \cdot (-1))^2 = (1-2)^2 = (-1)^2 = 1$$

From this it follows that the domain of f^{-1} is $x \geq 1$.