Section 2.6 Combining Functions

Sums, Differences, Products, and Quotients

Two functions f and g can be combined to form new functions $f+g$, $f-g$, fg, and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers.

Algebra of Functions

Let f and g be functions with domains A and B. Then the functions $f+g$, $f-g$, fg, and f/g are defined as follows.

$(f+g)(x) = f(x) + g(x)$ Domain $A \cap B$

$(f-g)(x) = f(x) - g(x)$ Domain $A \cap B$

$(fg)(x) = f(x)g(x)$ Domain $A \cap B$

$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ Domain $\{x \in A \cap B \mid g(x) \neq 0\}$

EXAMPLE: The domain of $f(x)=\sqrt{x}$ is $A=[0,\ \infty)$, the domain of $g(x)=\sqrt{1-x}$ is $B=(-\infty,\ 1]$, and the domain of $h(x)=\sqrt{x-1}$ is $C=[1,\infty)$, so the domain of

$$(f-g)(x)=\sqrt{x}-\sqrt{1-x} \quad \text{is} \quad A\cap B=[0,1]$$

and

$$(f-h)(x)=\sqrt{x}-\sqrt{x-1} \quad \text{is} \quad A\cap C=[1,\infty)$$

EXAMPLE: If $f(x)=x^2$ and $g(x)=x-1$, then the domain of the rational function

$$(f/g)(x)=x^2/(x-1) \quad \text{is} \quad \{x \mid x\neq 1\} \text{ or } (-\infty,\ 1)\cup(1,\ \infty)$$

Composition of Functions

There is another way of combining two functions to obtain a new function. For example, suppose that $y=f(u)=\sqrt{u}$ and $u=g(x)=x^2+1$. Since y is a function of u and u is, in turn, a function of x, it follows that y is ultimately a function of x. We compute this by substitution:

$$y=f(u)=f(g(x))=f(x^2+1)=\sqrt{x^2+1}$$

The procedure is called *composition* because the new function is *composed* of the two given functions f and g.

Composition of Functions

Given two functions f and g, the **composite function** $f\circ g$ (also called the **composition** of f and g) is defined by

$$(f\circ g)(x) = f(g(x))$$

EXAMPLE: If $f(x)=x^2+1$ and $g(x)=x-3$, find the following.

(a) $f\circ f$ (b) $f\circ g$ (c) $g\circ f$ (d) $g\circ g$ (e) $f(g(2))$ (f) $g(f(2))$

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Solution: We have

(a) $f\circ f=f(f(x))=\left\{\begin{array}{l} f(x^2+1) \\ \text{or} \\ (f(x))^2+1 \end{array}\right\}=(x^2+1)^2+1=(x^2)^2+2\cdot x^2\cdot 1+1^2+1=x^4+2x^2+2$

(b) $f\circ g=f(g(x))=\left\{\begin{array}{l} f(x-3) \\ \text{or} \\ (g(x))^2+1 \end{array}\right\}=(x-3)^2+1=x^2-2\cdot x\cdot 3+3^2+1=x^2-6x+10$

(c) $g\circ f=g(f(x))=\left\{\begin{array}{l} g(x^2+1) \\ \text{or} \\ f(x)-3 \end{array}\right\}=(x^2+1)-3=x^2-2$

(d) $g\circ g=g(g(x))=\left\{\begin{array}{l} g(x-3) \\ \text{or} \\ g(x)-3 \end{array}\right\}=(x-3)-3=x-6$

(e) $f(g(2))=(2-3)^2+1=(-1)^2+1=1+1=2$

(f) $g(f(2))=2^2-2=4-2=2$

EXAMPLE: If $f(x)=x$ and $g(x)=1$, then

$$f\circ f=x \qquad f\circ g=1 \qquad g\circ f=1 \qquad g\circ g=1$$

REMARK: You can see from the Examples above that sometimes $f\circ g=g\circ f$, but, in general, $f\circ g\neq g\circ f$.

The domain of $f\circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words, $(f\circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined.

EXAMPLE: If $f(x)=x^2$ and $g(x)=\sqrt{x}$, then

$$f\circ f=(x^2)^2=x^4 \qquad f\circ g=(\sqrt{x})^2=x,\ x\geq 0 \qquad g\circ f=\sqrt{x^2}=|x| \qquad g\circ g=\sqrt{\sqrt{x}}=\sqrt[4]{x}$$

(of course, the domain of $g\circ g=\sqrt[4]{x}$ is all nonnegative numbers).

EXAMPLE: If $f(x)=x^3$ and $g(x)=\sqrt[3]{x}$, then

$$f\circ f=(x^3)^3=x^9 \qquad f\circ g=(\sqrt[3]{x})^3=x \qquad g\circ f=\sqrt[3]{x^3}=x \qquad g\circ g=\sqrt[3]{\sqrt[3]{x}}=\sqrt[9]{x}$$

EXAMPLE: If $f(x)=\sqrt{x}$ and $g(x)=\sqrt{2-x}$, find each function and its domain.

(a) $f\circ g$ (b) $g\circ f$ (c) $f\circ f$ (d) $g\circ g$

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Solution:

(a) We have

$$(f\circ g)(x)=f(g(x))=f(\sqrt{2-x})=\sqrt{\sqrt{2-x}}=\sqrt[4]{2-x}$$

The domain of $f\circ g$ is $\{x \mid 2-x\geq 0\}=\{x \mid x\leq 2\}=(-\infty,\ 2]$.

(b) We have

$$(g\circ f)(x)=g(f(x))=g(\sqrt{x})=\sqrt{2-\sqrt{x}}$$

For $\sqrt{x}$ to be defined we must have $x\geq 0$. For $\sqrt{2-\sqrt{x}}$ to be defined we must have $2-\sqrt{x}\geq 0$, that is, $\sqrt{x}\leq 2$, or $x\leq 4$. Thus we have $0\leq x\leq 4$, so the domain of $g\circ f$ is the closed interval $[0,4]$.

(c) We have

$$(f\circ f)(x)=f(f(x))=f(\sqrt{x})=\sqrt{\sqrt{x}}=\sqrt[4]{x}$$

The domain of $f\circ f$ is $[0,\ \infty)$.

(d) We have

$$(g\circ g)(x)=g(g(x))=g(\sqrt{2-x})=\sqrt{2-\sqrt{2-x}}$$

This expression is defined when both $2-x\geq 0$ and $2-\sqrt{2-x}\geq 0$. The first inequality means $x\leq 2$, and the second is equivalent to $\sqrt{2-x}\leq 2$, or $2-x\leq 4$, or $x\geq -2$. Thus $-2\leq x\leq 2$, so the domain of $g\circ g$ is the closed interval [-2, 2].

It is possible to take the composition of three or more functions. For instance, the composite function $f\circ g\circ h$ is found by first applying h, then g, and then f as follows:

$$(f\circ g\circ h)(x)=f(g(h(x)))$$

EXAMPLE: Find $f\circ g\circ h$ if $f(x)=x/(x+1)$, $g(x)=x^{10}$, and $h(x)=x+3$.

Solution: We have

$$(f\circ g\circ h)(x)=f(g(h(x)))=f(g(x+3))=f((x+3)^{10})=\frac{(x+3)^{10}}{(x+3)^{10}+1}$$

So far we have used composition to build complicated functions from simpler ones. But in calculus it is often useful to be able to *decompose* a complicated function into simpler ones, as in the following example.

EXAMPLE: Given $F(x)=\dfrac{1}{(x+9)^2}$, find functions f,g, and h such that $F=f\circ g\circ h$.

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Solution 1: The formula for F says: First add 9, then square $x+9$, and finally divide 1 by the result. So we let

$$f(x)=\frac{1}{x},\quad g(x)=x^2,\quad h(x)=x+9$$

Then

$$(f\circ g\circ h)(x)=f(g(h(x)))=f(g(x+9))=f((x+9)^2)=\frac{1}{(x+9)^2}=F(x)$$

Solution 2: Here is an other way to look at F: First add 9, then divide 1 by $x+9$, and finally square the result. So we let

$$f(x)=x^2,\quad g(x)=\frac{1}{x},\quad h(x)=x+9$$

Then

$$(f\circ g\circ h)(x)=f(g(h(x)))=f(g(x+9))=f\left(\frac{1}{x+9}\right)=\left(\frac{1}{x+9}\right)^2=\frac{1}{(x+9)^2}=F(x)$$