Section 2.6 Combining Functions
Sums, Differences, Products, and Quotients

Two functions \( f \) and \( g \) can be combined to form new functions \( f + g, f - g, fg, \) and \( f/g \) in a manner similar to the way we add, subtract, multiply, and divide real numbers.

**Algebra of Functions**

Let \( f \) and \( g \) be functions with domains \( A \) and \( B \). Then the functions \( f + g, f - g, fg, \) and \( f/g \) are defined as follows.

\[
\begin{align*}
(f + g)(x) &= f(x) + g(x) & \text{Domain } A \cap B \\
(f - g)(x) &= f(x) - g(x) & \text{Domain } A \cap B \\
(fg)(x) &= f(x)g(x) & \text{Domain } A \cap B \\
\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} \quad \text{Domain } \{x \in A \cap B \mid g(x) \neq 0\}
\end{align*}
\]

**EXAMPLE:** The domain of \( f(x) = \sqrt{x} \) is \( A = [0, \infty) \), the domain of \( g(x) = \sqrt{1-x} \) is \( B = (-\infty, 1] \), and the domain of \( h(x) = \sqrt{x-1} \) is \( C = [1, \infty) \), so the domain of

\[
(f - g)(x) = \sqrt{x} - \sqrt{1-x} \quad \text{is} \quad A \cap B = [0, 1]
\]

and

\[
(f - h)(x) = \sqrt{x} - \sqrt{x-1} \quad \text{is} \quad A \cap C = [1, \infty)
\]

**EXAMPLE:** If \( f(x) = x^2 \) and \( g(x) = x - 1 \), then the domain of the rational function

\[
(f/g)(x) = \frac{x^2}{x - 1} \quad \text{is} \quad \{x \mid x \neq 1\} \text{ or } (-\infty, 1) \cup (1, \infty)
\]

**Composition of Functions**

There is another way of combining two functions to obtain a new function. For example, suppose that \( y = f(u) = \sqrt{u} \) and \( u = g(x) = x^2 + 1 \). Since \( y \) is a function of \( u \) and \( u \) is, in turn, a function of \( x \), it follows that \( y \) is ultimately a function of \( x \). We compute this by substitution:

\[
y = f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}
\]

The procedure is called composition because the new function is composed of the two given functions \( f \) and \( g \).

**Composition of Functions**

Given two functions \( f \) and \( g \), the **composite function** \( f \circ g \) (also called the composition of \( f \) and \( g \)) is defined by

\[
(f \circ g)(x) = f(g(x))
\]

**EXAMPLE:** If \( f(x) = x^2 + 1 \) and \( g(x) = x - 3 \), find the following.

(a) \( f \circ f \) \hspace{1cm} (b) \( f \circ g \) \hspace{1cm} (c) \( g \circ f \) \hspace{1cm} (d) \( g \circ g \) \hspace{1cm} (e) \( f(g(2)) \) \hspace{1cm} (f) \( g(f(2)) \)
EXAMPLE: If \( f(x) = x^2 + 1 \) and \( g(x) = x - 3 \), find the following.

(a) \( f \circ f \)  
(b) \( f \circ g \)  
(c) \( g \circ f \)  
(d) \( g \circ g \)  
(e) \( f(g(2)) \)  
(f) \( g(f(2)) \)

Solution: We have

(a) \( f \circ f = f(f(x)) = \begin{cases} f(x^2 + 1) \\ (f(x))^2 + 1 \end{cases} = (x^2 + 1)^2 + 1 = (x^2)^2 + 2x^2 \cdot 1 + 1^2 + 1 = x^4 + 2x^2 + 2 \)

(b) \( f \circ g = f(g(x)) = \begin{cases} f(x - 3) \\ (g(x))^2 + 1 \end{cases} = (x - 3)^2 + 1 = x^2 - 2x \cdot 3 + 3^2 + 1 = x^2 - 6x + 10 \)

(c) \( g \circ f = g(f(x)) = \begin{cases} g(x^2 + 1) \\ f(x) - 3 \end{cases} = (x^2 + 1) - 3 = x^2 - 2 \)

(d) \( g \circ g = g(g(x)) = \begin{cases} g(x - 3) \\ g(x) - 3 \end{cases} = (x - 3) - 3 = x - 6 \)

(e) \( f(g(2)) = (2 - 3)^2 + 1 = (-1)^2 + 1 = 1 + 1 = 2 \)

(f) \( g(f(2)) = 2^2 - 2 = 4 - 2 = 2 \)

EXAMPLE: If \( f(x) = x \) and \( g(x) = 1 \), then

\[ f \circ f = x \quad f \circ g = 1 \quad g \circ f = 1 \quad g \circ g = 1 \]

REMARK: You can see from the Examples above that sometimes \( f \circ g = g \circ f \), but, in general, \( f \circ g \neq g \circ f \).

The domain of \( f \circ g \) is the set of all \( x \) in the domain of \( g \) such that \( g(x) \) is in the domain of \( f \). In other words, \( (f \circ g)(x) \) is defined whenever both \( g(x) \) and \( f(g(x)) \) are defined.

EXAMPLE: If \( f(x) = x^2 \) and \( g(x) = \sqrt{x} \), then

\[ f \circ f = (x^2)^2 = x^4 \quad f \circ g = (\sqrt{x})^2 = x, \quad x \geq 0 \quad g \circ f = \sqrt{x^2} = |x| \quad g \circ g = \sqrt{\sqrt{x}} = \sqrt[4]{x} \]

(of course, the domain of \( g \circ g = \sqrt[4]{x} \) is all nonnegative numbers).

EXAMPLE: If \( f(x) = x^3 \) and \( g(x) = \sqrt[3]{x} \), then

\[ f \circ f = (x^3)^3 = x^9 \quad f \circ g = (\sqrt[3]{x})^3 = x \quad g \circ f = \sqrt[3]{x^3} = x \quad g \circ g = \sqrt[3]{\sqrt[3]{x}} = \sqrt[9]{x} \]

EXAMPLE: If \( f(x) = \sqrt{x} \) and \( g(x) = \sqrt{2 - x} \), find each function and its domain.

(a) \( f \circ g \)  
(b) \( g \circ f \)  
(c) \( f \circ f \)  
(d) \( g \circ g \)
EXAMPLE: If \( f(x) = \sqrt{x} \) and \( g(x) = \sqrt{2-x} \), find each function and its domain.

(a) \( f \circ g \) \quad (b) \( g \circ f \) \quad (c) \( f \circ f \) \quad (d) \( g \circ g \)

Solution:

(a) We have
\[
(f \circ g)(x) = f(g(x)) = f(\sqrt{2-x}) = \sqrt{\sqrt{2-x}} = \sqrt{2-x}
\]
The domain of \( f \circ g \) is \( \{x \mid 2-x \geq 0\} = \{x \mid x \leq 2\} = (-\infty, 2] \).

(b) We have
\[
(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{2-\sqrt{x}}
\]
For \( \sqrt{x} \) to be defined we must have \( x \geq 0 \). For \( \sqrt{2-\sqrt{x}} \) to be defined we must have \( 2-\sqrt{x} \geq 0 \), that is, \( \sqrt{x} \leq 2 \), or \( x \leq 4 \). Thus we have \( 0 \leq x \leq 4 \), so the domain of \( g \circ f \) is the closed interval \([0, 4]\).

(c) We have
\[
(f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt{x}
\]
The domain of \( f \circ f \) is \([0, \infty)\).

(d) We have
\[
(g \circ g)(x) = g(g(x)) = g(\sqrt{2-x}) = \sqrt{2-\sqrt{2-x}}
\]
This expression is defined when both \( 2-x \geq 0 \) and \( 2-\sqrt{2-x} \geq 0 \). The first inequality means \( x \leq 2 \), and the second is equivalent to \( \sqrt{2-x} \leq 2 \), or \( 2-x \leq 4 \), or \( x \geq -2 \). Thus \( -2 \leq x \leq 2 \), so the domain of \( g \circ g \) is the closed interval \([-2, 2]\).

It is possible to take the composition of three or more functions. For instance, the composite function \( f \circ g \circ h \) is found by first applying \( h \), then \( g \), and then \( f \) as follows:
\[
(f \circ g \circ h)(x) = f(g(h(x)))
\]

EXAMPLE: Find \( f \circ g \circ h \) if \( f(x) = x/(x+1) \), \( g(x) = x^{10} \), and \( h(x) = x + 3 \).

Solution: We have
\[
(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x+3)) = f((x+3)^{10}) = \frac{(x+3)^{10}}{(x+3)^{10}+1}
\]

So far we have used composition to build complicated functions from simpler ones. But in calculus it is often useful to be able to decompose a complicated function into simpler ones, as in the following example.

EXAMPLE: Given \( F(x) = \frac{4}{(x+9)^2} \), find functions \( f, g, \) and \( h \) such that \( F = f \circ g \circ h \).
EXAMPLE: Given $F(x) = \frac{4}{(x+9)^2}$, find functions $f, g,$ and $h$ such that $F = f \circ g \circ h$.

Solution 1: The formula for $F$ says: First add 9, then square $x + 9$, and finally divide 4 by the result. So we let

$$f(x) = \frac{4}{x}, \quad g(x) = x^2, \quad h(x) = x + 9$$

Then

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x + 9)) = f((x + 9)^2) = \frac{4}{(x + 9)^2} = F(x)$$

Solution 2: Here is another way to look at $F$: First add 9, then divide 2 by $x + 9$, and finally square the result. So we let

$$f(x) = x^2, \quad g(x) = \frac{2}{x}, \quad h(x) = x + 9$$

Then

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x + 9)) = f\left(\frac{2}{x + 9}\right) = \left(\frac{2}{x + 9}\right)^2 = \frac{4}{(x + 9)^2} = F(x)$$