

Section 2.6 Combining Functions

Sums, Differences, Products, and Quotients

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg , and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers.

Algebra of Functions

Let f and g be functions with domains A and B . Then the functions $f + g$, $f - g$, fg , and f/g are defined as follows.

$$(f + g)(x) = f(x) + g(x) \quad \text{Domain } A \cap B$$

$$(f - g)(x) = f(x) - g(x) \quad \text{Domain } A \cap B$$

$$(fg)(x) = f(x)g(x) \quad \text{Domain } A \cap B$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \text{Domain } \{x \in A \cap B \mid g(x) \neq 0\}$$

EXAMPLE: The domain of $f(x) = \sqrt{x}$ is $A = [0, \infty)$, the domain of $g(x) = \sqrt{1-x}$ is $B = (-\infty, 1]$, and the domain of $h(x) = \sqrt{x-1}$ is $C = [1, \infty)$, so the domain of

$$(f - g)(x) = \sqrt{x} - \sqrt{1-x} \quad \text{is } A \cap B = [0, 1]$$

and

$$(f - h)(x) = \sqrt{x} - \sqrt{x-1} \quad \text{is } A \cap C = [1, \infty)$$

EXAMPLE: If $f(x) = x^2$ and $g(x) = x - 1$, then the domain of the rational function

$$(f/g)(x) = x^2/(x-1) \quad \text{is } \{x \mid x \neq 1\} \text{ or } (-\infty, 1) \cup (1, \infty)$$

Composition of Functions

There is another way of combining two functions to obtain a new function. For example, suppose that $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2 + 1$. Since y is a function of u and u is, in turn, a function of x , it follows that y is ultimately a function of x . We compute this by substitution:

$$y = f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The procedure is called *composition* because the new function is *composed* of the two given functions f and g .

Composition of Functions

Given two functions f and g , the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

EXAMPLE: If $f(x) = x^2 + 1$ and $g(x) = x - 3$, find the following.

- (a) $f \circ f$ (b) $f \circ g$ (c) $g \circ f$ (d) $g \circ g$ (e) $f(g(2))$ (f) $g(f(2))$

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Solution: We have

$$(a) f \circ f = f(f(x)) = \left\{ \begin{array}{l} f(x^2 + 1) \\ \text{or} \\ (f(x))^2 + 1 \end{array} \right\} = (x^2 + 1)^2 + 1 = (x^2)^2 + 2 \cdot x^2 \cdot 1 + 1^2 + 1 = x^4 + 2x^2 + 2$$

$$(b) f \circ g = f(g(x)) = \left\{ \begin{array}{l} f(x - 3) \\ \text{or} \\ (g(x))^2 + 1 \end{array} \right\} = (x - 3)^2 + 1 = x^2 - 2 \cdot x \cdot 3 + 3^2 + 1 = x^2 - 6x + 10$$

$$(c) g \circ f = g(f(x)) = \left\{ \begin{array}{l} g(x^2 + 1) \\ \text{or} \\ f(x) - 3 \end{array} \right\} = (x^2 + 1) - 3 = x^2 - 2$$

$$(d) g \circ g = g(g(x)) = \left\{ \begin{array}{l} g(x - 3) \\ \text{or} \\ g(x) - 3 \end{array} \right\} = (x - 3) - 3 = x - 6$$

$$(e) f(g(2)) = (2 - 3)^2 + 1 = (-1)^2 + 1 = 1 + 1 = 2$$

$$(f) g(f(2)) = 2^2 - 2 = 4 - 2 = 2$$

EXAMPLE: If $f(x) = x$ and $g(x) = 1$, then

$$f \circ f = x \qquad f \circ g = 1 \qquad g \circ f = 1 \qquad g \circ g = 1$$

REMARK: You can see from the Examples above that sometimes $f \circ g = g \circ f$, but, in general, $f \circ g \neq g \circ f$.

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f . In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined.

EXAMPLE: If $f(x) = x^2$ and $g(x) = \sqrt{x}$, then

$$f \circ f = (x^2)^2 = x^4 \qquad f \circ g = (\sqrt{x})^2 = x, \quad x \geq 0 \qquad g \circ f = \sqrt{x^2} = |x| \qquad g \circ g = \sqrt{\sqrt{x}} = \sqrt[4]{x}$$

(of course, the domain of $g \circ g = \sqrt[4]{x}$ is all nonnegative numbers).

EXAMPLE: If $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$, then

$$f \circ f = (x^3)^3 = x^9 \qquad f \circ g = (\sqrt[3]{x})^3 = x \qquad g \circ f = \sqrt[3]{x^3} = x \qquad g \circ g = \sqrt[3]{\sqrt[3]{x}} = \sqrt[9]{x}$$

EXAMPLE: If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2 - x}$, find each function and its domain.

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Solution:

(a) We have

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{2-x}) = \sqrt{\sqrt{2-x}} = \sqrt[4]{2-x}$$

The domain of $f \circ g$ is $\{x \mid 2-x \geq 0\} = \{x \mid x \leq 2\} = (-\infty, 2]$.

(b) We have

$$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{2-\sqrt{x}}$$

For \sqrt{x} to be defined we must have $x \geq 0$. For $\sqrt{2-\sqrt{x}}$ to be defined we must have $2-\sqrt{x} \geq 0$, that is, $\sqrt{x} \leq 2$, or $x \leq 4$. Thus we have $0 \leq x \leq 4$, so the domain of $g \circ f$ is the closed interval $[0, 4]$.

(c) We have

$$(f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$$

The domain of $f \circ f$ is $[0, \infty)$.

(d) We have

$$(g \circ g)(x) = g(g(x)) = g(\sqrt{2-x}) = \sqrt{2-\sqrt{2-x}}$$

This expression is defined when both $2-x \geq 0$ and $2-\sqrt{2-x} \geq 0$. The first inequality means $x \leq 2$, and the second is equivalent to $\sqrt{2-x} \leq 2$, or $2-x \leq 4$, or $x \geq -2$. Thus $-2 \leq x \leq 2$, so the domain of $g \circ g$ is the closed interval $[-2, 2]$.

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h , then g , and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

EXAMPLE: Find $f \circ g \circ h$ if $f(x) = x/(x+1)$, $g(x) = x^{10}$, and $h(x) = x+3$.

Solution: We have

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x+3)) = f((x+3)^{10}) = \frac{(x+3)^{10}}{(x+3)^{10}+1}$$

So far we have used composition to build complicated functions from simpler ones. But in calculus it is often useful to be able to *decompose* a complicated function into simpler ones, as in the following example.

EXAMPLE: Given $F(x) = \frac{4}{(x+9)^2}$, find functions f, g , and h such that $F = f \circ g \circ h$.

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Solution 1: The formula for F says: First add 9, then square $x + 9$, and finally divide 4 by the result. So we let

$$f(x) = \frac{4}{x}, \quad g(x) = x^2, \quad h(x) = x + 9$$

Then

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x + 9)) = f((x + 9)^2) = \frac{4}{(x + 9)^2} = F(x)$$

Solution 2: Here is an other way to look at F : First add 9, then divide 2 by $x + 9$, and finally square the result. So we let

$$f(x) = x^2, \quad g(x) = \frac{2}{x}, \quad h(x) = x + 9$$

Then

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x + 9)) = f\left(\frac{2}{x + 9}\right) = \left(\frac{2}{x + 9}\right)^2 = \frac{4}{(x + 9)^2} = F(x)$$