Section 2.1 What Is a Function?

DEFINITION: A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B. The set A is called the **domain** of f. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

EXAMPLES:

1. Let $f(x) = x + \sqrt{x}$. Then

$$\begin{aligned} f(0) &= 0 + \sqrt{0} = 0 \\ f(4) &= 4 + \sqrt{4} = 6 \end{aligned}$$

2. Let $f(x) = 3x^2 + x - 5$. Then

$$\begin{aligned} f(-2) &= 3 \cdot (-2)^2 + (-2) - 5 = 5 \\ f(0) &= 3 \cdot 0^2 + 0 - 5 = -5 \\ f(4) &= 3 \cdot 4^2 + 4 - 5 = 47 \\ f\left(\frac{1}{2}\right) &= 3 \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{2} - 5 = -\frac{15}{4} \end{aligned}$$

3. Let $f(x) = \dfrac{\sqrt{x+1}}{x}$. Then

$$f(3) = \frac{\sqrt{3+1}}{3} = \frac{\sqrt{4}}{3} = \frac{2}{3} \qquad f(5) = \frac{\sqrt{5+1}}{5} = \frac{\sqrt{6}}{5}$$

and

$$f(a-1) = \frac{\sqrt{a-1+1}}{a-1} = \frac{\sqrt{a}}{a-1}$$

4. Let

$$f(x) = \begin{cases} 3x^2 + x - 5 & \text{if } x \le 0 \\ x + \sqrt{x} & \text{if } x > 0 \end{cases}$$

and

$$g(x) = \begin{cases} 3x^2 + x - 5 & \text{if } x < 0 \\ x + \sqrt{x} & \text{if } x \ge 0 \end{cases}$$

Then

$$\begin{aligned} f(-2) &= 3 \cdot (-2)^2 + (-2) - 5 = 5 \\ f(0) &= 3 \cdot 0^2 + 0 - 5 = -5 \\ f(4) &= 4 + \sqrt{4} = 6 \end{aligned}$$

and

$$\begin{aligned} g(-2) &= 3 \cdot (-2)^2 + (-2) - 5 = 5 \\ g(0) &= 0 + \sqrt{0} = 0 \\ g(4) &= 4 + \sqrt{4} = 6 \end{aligned}$$

Four Ways to Represent a Function

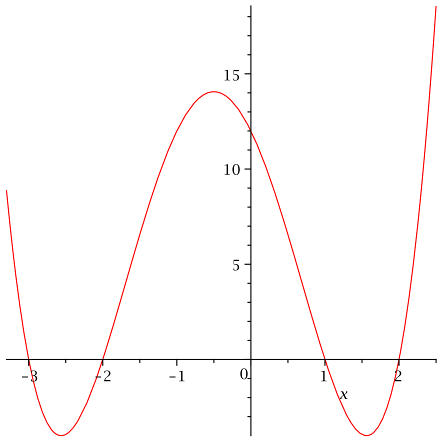
There are 4 possible ways to represent a function:

- Verbally (in words)
- Visually (by a graph)
- Numerically (by a table of values)
- Algebraically (by an explicit formula)

EXAMPLES:

1. Verbally: **“$s(t)$ is speed of a car at time t”**

2. Visually:



3. Numerically:

x	$f(x)$
1	2
2	8
7	-1
10	5

4. Algebraically:

(a) $f(x) = 1,\ g(x) = \dfrac{x}{x},\ h(x) = x^2,\ F(x) = \dfrac{1}{x},\ y = \dfrac{5\sin x - 4}{\ln x}$

(b) $f(x) = \dfrac{1}{x},\ x > 5$

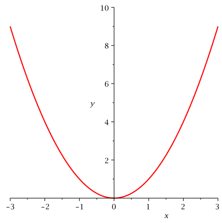
(c) $f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ x+1 & \text{if } x > 2 \end{cases}$

Domain and Range

EXAMPLES:

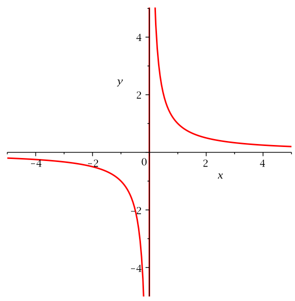
1. $f(x) = x^2$

Domain: All real numbers or $(-\infty, \infty)$.
Range: $\{y \mid y \geq 0\}$ or $[0, \infty)$.



2. $f(x) = \dfrac{1}{x}$

Domain: $\{x \mid x \neq 0\}$ or $(-\infty, 0) \cup (0, \infty)$, since $x \neq 0$.
Range: $\{y \mid y \neq 0\}$ or $(-\infty, 0) \cup (0, \infty)$.

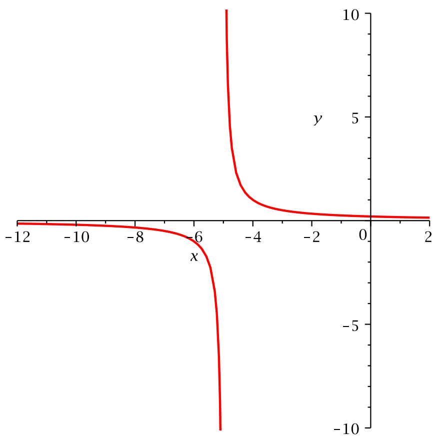


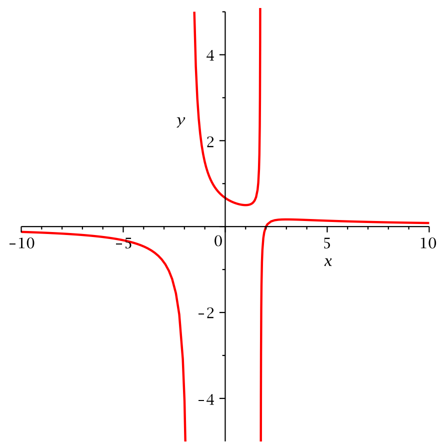
3. $f(x) = \dfrac{1}{x}, \ x \geq 2$

Domain: $\{x \mid x \geq 2\}$ or $[2, \infty)$, since $x \geq 2$.
Range: $\{y \mid 0 < y \leq 1/2\}$ or $(0, 1/2]$.

4. $f(x) = \dfrac{1}{x+5}$

Domain: $\{x \mid x \neq -5\}$ or $(-\infty, -5) \cup (-5, \infty)$, since $x + 5 \neq 0$.
Range: $\{y \mid y \neq 0\}$ or $(-\infty, 0) \cup (0, \infty)$.





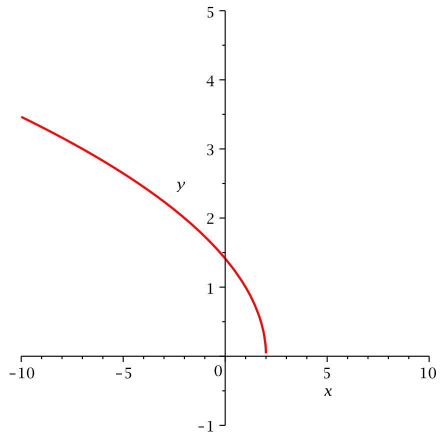
5. $f(x) = \dfrac{x-2}{x^2-3}$

Domain: $\{x \mid x \neq \pm\sqrt{3}\}$ or $(-\infty, -\sqrt{3}) \cup (-\sqrt{3}, \sqrt{3}) \cup (\sqrt{3}, \infty)$, since $x^2 - 3 \neq 0$.
Range: $\left(-\infty, \dfrac{1}{6}\right] \cup \left[\dfrac{1}{2}, \infty\right)$.

6. $f(x) = \sqrt{2-x}$

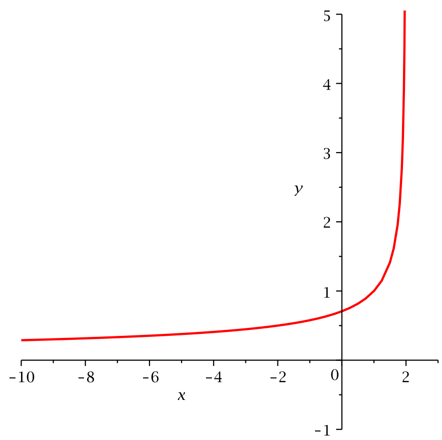
6. $f(x) = \sqrt{2-x}$

Domain: $\{x \mid x \le 2\}$ or $(-\infty, 2]$, since $2 - x \ge 0$.
Range: $\{y \mid y \ge 0\}$ or $[0, \infty)$.



7. $f(x) = \dfrac{1}{\sqrt{2-x}}$

Domain: $\{x \mid x < 2\}$ or $(-\infty, 2)$, since $2 - x > 0$.
Range: $\{y \mid y > 0\}$ or $(0, \infty)$.



8. $f(x) = \dfrac{1}{\sqrt{2x - x^2}}$

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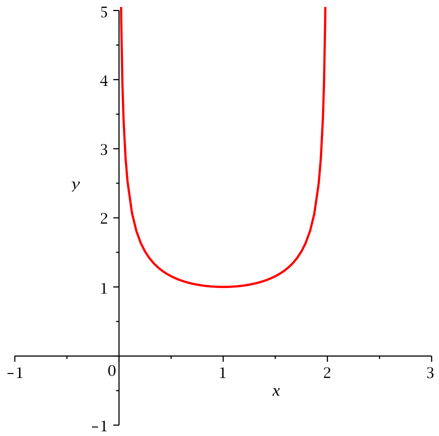
Domain: $\{x \mid 0 < x < 2\}$ or $(0, 2)$, since $2x - x^2 = x(2 - x) > 0$.
Range: $\{y \mid y \geq 1\}$ or $[1, \infty)$.

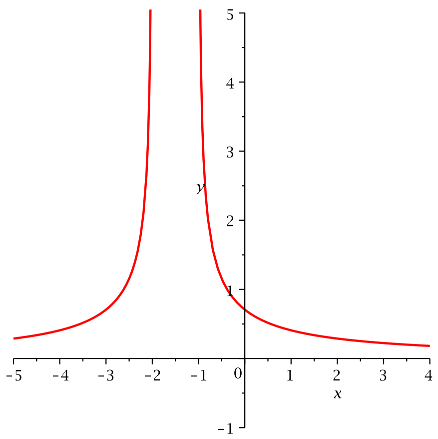


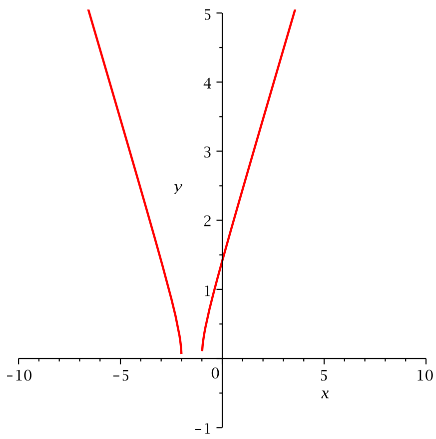
9. $f(x) = \dfrac{1}{\sqrt{x^2 + 3x + 2}}$

Domain: $\{x \mid x < -2 \text{ or } x > -1\}$ or $(-\infty, -2) \cup (-1, \infty)$, since $x^2 + 3x + 2 = (x + 1)(x + 2) > 0$.
Range: $\{y \mid y > 0\}$ or $(0, \infty)$.

10. $f(x) = \sqrt{x^2 + 3x + 2}$

Domain: $\{x \mid x \leq -2 \text{ or } x \geq -1\}$ or $(-\infty, -2] \cup [-1, \infty)$, since $x^2 + 3x + 2 = (x + 1)(x + 2) \geq 0$.
Range: $\{y \mid y \geq 0\}$ or $[0, \infty)$.

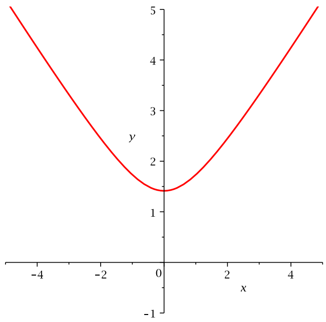




11. $f(x) = \sqrt{x^2 + 2}$

11. $f(x) = \sqrt{x^2 + 2}$

Domain: All real numbers, or $(-\infty, \infty)$, since $x^2 + 2$ is always > 0.
Range: $\{y \mid y \geq \sqrt{2}\}$ or $[\sqrt{2}, \infty)$.

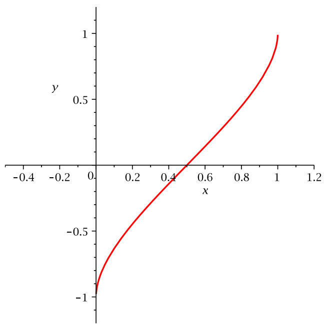


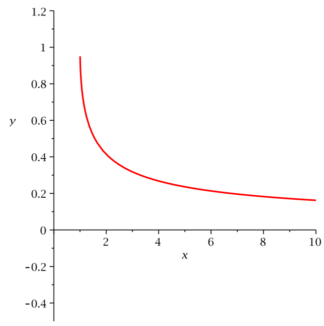
12. $f(x) = \sqrt{x} - \sqrt{1-x}$

Domain: $\{x \mid 0 \leq x \leq 1\}$ or $[0, 1]$, since $x \geq 0$ and $1 - x \geq 0$.
Range: $\{y \mid -1 \leq y \leq 1\}$ or $[-1, 1]$.

13. $f(x) = \sqrt{x} - \sqrt{x-1}$

Domain: $\{x \mid x \geq 1\}$ or $[1, \infty)$, since $x \geq 0$ and $x - 1 \geq 0$.
Range: $\{y \mid 0 < y \leq 1\}$ or $(0, 1]$.





14. $f(x) = x^6 + x^2 + x - 1$

Domain: $(-\infty, \infty)$.
Range: $\{y \mid y \geq -1.2392\}$ or $[-1.2392, \infty)$.