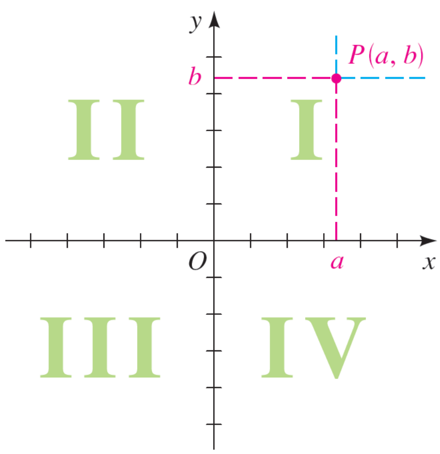
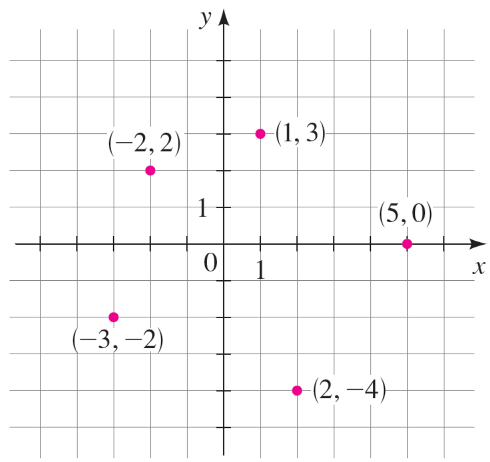
Section 1.8 Coordinate Geometry

The Coordinate Plane

Just as points on a line can be identified with real numbers to form the coordinate line, points in a plane can be identified with ordered pairs of numbers to form the **coordinate plane** or **Cartesian plane**. To do this, we draw two perpendicular real lines that intersect at 0 on each line. Usually, one line is horizontal with positive direction to the right and is called the ***x*-axis**; the other line is vertical with positive direction upward and is called the ***y*-axis**. The point of intersection of the x-axis and the y-axis is the **origin *O*,** and the two axes divide the plane into four **quadrants**, labeled I, II, III, and IV in the Figure below (left). (The points *on* the coordinate axes are not assigned to any quadrant.)





Any point P in the coordinate plane can be located by a unique **ordered pair** of numbers (a, b), as shown in the Figure above (left). The first number a is called the ***x*-coordinate** of P; the second number b is called the ***y*-coordinate** of P. We can think of the coordinates of P as its "address," because they specify its location in the plane. Several points are labeled with their coordinates in the Figure above (right).

EXAMPLE: Describe and sketch the regions given by each set.

(a) $\{(x, y) \mid x \geq 0\}$ (b) $\{(x, y) \mid y = 1\}$ (c) $\{(x, y) \mid |y| < 1\}$

EXAMPLE: Describe and sketch the regions given by each set.

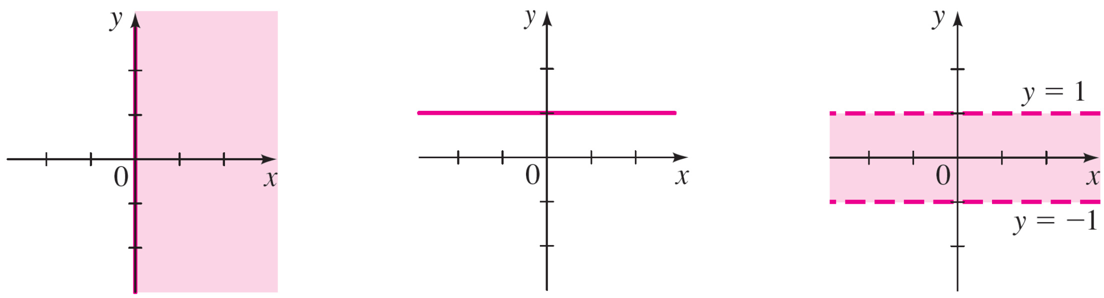
(a) $\{(x,y) \mid x \geq 0\}$ (b) $\{(x,y) \mid y = 1\}$ (c) $\{(x,y) \mid |y| < 1\}$

Solution:

(a) The points whose x-coordinates are 0 or positive lie on the y-axis or to the right of it.

(b) The set of all points with y-coordinate 1 is a horizontal line one unit above the x-axis.

(c) The region consists of all points that lie between (but not on) the horizontal lines $y = 1$ and $y = -1$.



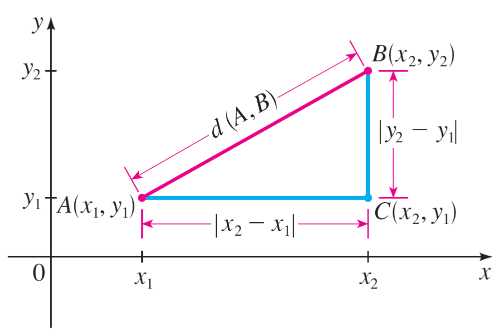
(a) $x \geq 0$ (b) $y = 1$ (c) $|y| < 1$

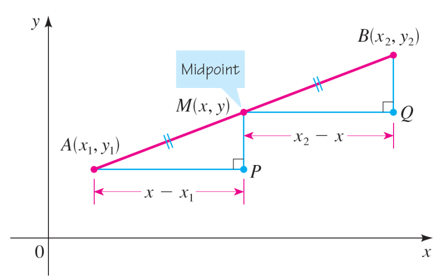
The Distance and Midpoint Formulas

Distance Formula

The distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the plane is

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$





Midpoint Formula

The midpoint of the line segment from $A(x_1, y_1)$ to $B(x_2, y_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Graphs of Equations in Two Variables

An **equation in two variables**, such as $y = x^2 + 1$, expresses a relationship between two quantities. A point (x, y) **satisfies** the equation if it makes the equation true when the values for x and y are substituted into the equation. For example, the point $(3, 10)$ satisfies the equation $y = x^2 + 1$ because $10 = 3^2 + 1$, but the point $(1, 3)$ does not, because $3 \neq 1^2 + 1$.

The Graph of an Equation

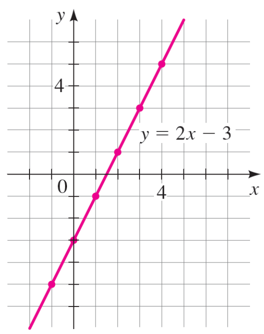
The **graph** of an equation in x and y is the set of all points (x, y) in the coordinate plane that satisfy the equation.

EXAMPLE: Sketch the graph of the equation $2x - y = 3$.

Solution: We first solve the given equation for y to get

$$y = 2x - 3$$

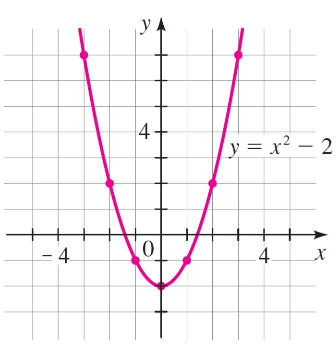
This helps us calculate the y-coordinates in the following table and eventually sketch the graph.



x	$y = 2x - 3$	(x, y)
-1	-5	$(-1, -5)$
0	-3	$(0, -3)$
1	-1	$(1, -1)$
2	1	$(2, 1)$
3	3	$(3, 3)$
4	5	$(4, 5)$

EXAMPLE: Sketch the graph of the equation $y = x^2 - 2$.

Solution: We find some of the points that satisfy the equation in the following table. In the Figure below we plot these points and then connect them by a smooth curve. A curve with this shape is called a *parabola*.



x	$y = x^2 - 2$	(x, y)
-3	7	$(-3, 7)$
-2	2	$(-2, 2)$
-1	-1	$(-1, -1)$
0	-2	$(0, -2)$
1	-1	$(1, -1)$
2	2	$(2, 2)$
3	7	$(3, 7)$

EXAMPLE: Sketch the graph of the equation $y = |x|$.

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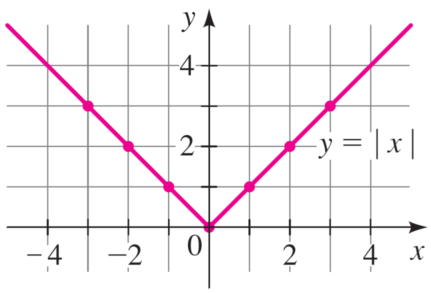
Solution: We make a table of values that helps us to sketch the graph of the equation.

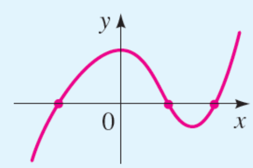
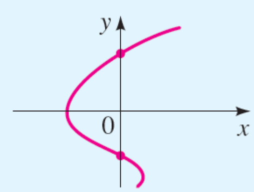


x	$y = \|x\|$	(x, y)
−3	3	(−3, 3)
−2	2	(−2, 2)
−1	1	(−1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	2	(2, 2)
3	3	(3, 3)

Intercepts

The x-coordinates of the points where a graph intersects the x-axis are called the ***x*-intercepts** of the graph and are obtained by setting $y = 0$ in the equation of the graph. The y-coordinates of the points where a graph intersects the y-axis are called the ***y*-intercepts** of the graph and are obtained by setting $x = 0$ in the equation of the graph.

Definition of Intercepts

Intercepts	How to find them	Where they are on the graph
***x*-intercepts:** The x-coordinates of points where the graph of an equation intersects the x-axis	Set $y = 0$ and solve for x	
***y*-intercepts:** The y-coordinates of points where the graph of an equation intersects the y-axis	Set $x = 0$ and solve for y	

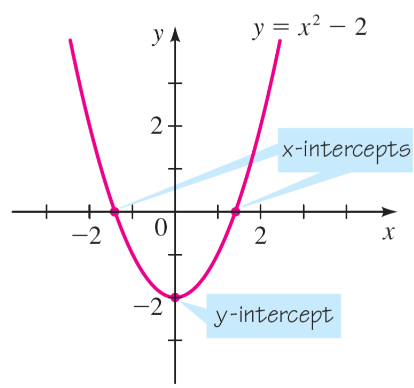
EXAMPLE: Find the x- and y-intercepts of the graph of the equation $y = x^2 - 2$.

Solution: To find the x-intercepts, we set $y = 0$ and solve for x. Thus

$$0 = x^2 - 2$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$



The x-intercepts are $\sqrt{2}$ and $-\sqrt{2}$.

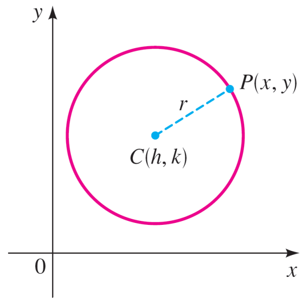
To find the y-intercepts, we set $x = 0$ and solve for y. Thus

$$y = 0^2 - 2$$

$$y = -2$$

The y-intercept is -2.

Circles



Equation of a Circle

An equation of the circle with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

This is called the **standard form** for the equation of the circle. If the center of the circle is the origin $(0, 0)$, then the equation is

$$x^2 + y^2 = r^2$$

EXAMPLE: Graph each equation.

(a) $x^2 + y^2 = 25$ (b) $(x - 2)^2 + (y + 1)^2 = 25$

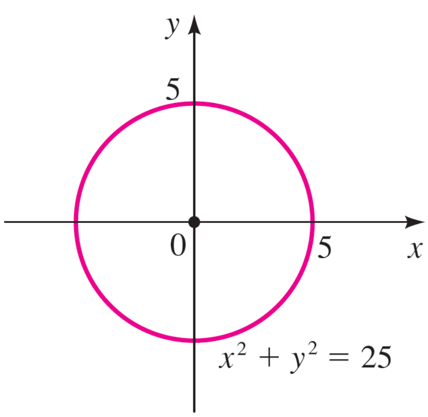
Solution:

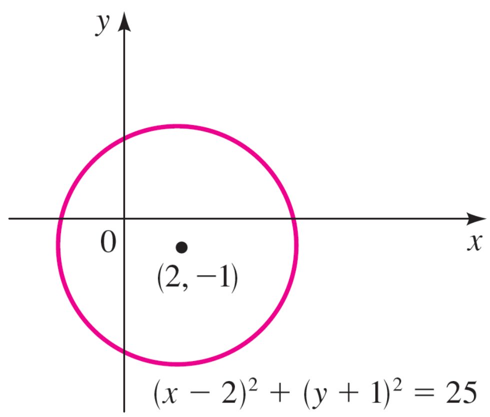
(a) Rewriting the equation as $x^2 + y^2 = 5^2$, we see that this is an equation of the circle of radius 5 centered at the origin.

(b) Rewriting the equation as

$$(x - 2)^2 + (y - (-1))^2 = 5^2$$

we see that this is an equation of the circle of radius 5 centered at $(2, -1)$.





EXAMPLE:

(a) Find an equation of the circle with radius 3 and center $(2, -5)$.

(b) Find an equation of the circle that has the points $P(1, 8)$ and $Q(5, -6)$ as the endpoints of a diameter.

EXAMPLE:

(a) Find an equation of the circle with radius 3 and center $(2, -5)$.

(b) Find an equation of the circle that has the points $P(1, 8)$ and $Q(5, -6)$ as the endpoints of a diameter.

Solution:

(a) Using the equation of a circle with $r = 3$, $h = 2$, and $k = -5$, we obtain

$$(x-2)^2 + (y-(-5))^2 = 3^2 \quad \Longrightarrow \quad (x-2)^2 + (y+5)^2 = 9$$

(b) We first observe that the center is the midpoint of the diameter PQ, so by the Midpoint Formula the center is

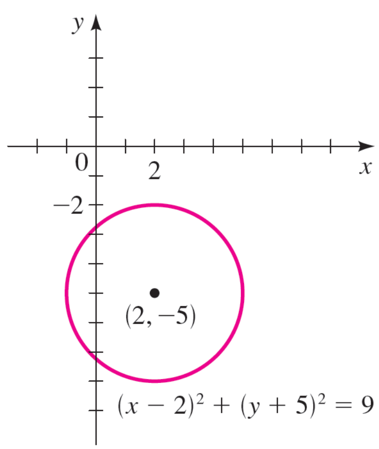
$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{1+5}{2}, \frac{8+(-6)}{2}\right) = \left(\frac{6}{2}, \frac{2}{2}\right) = (3, 1)$$

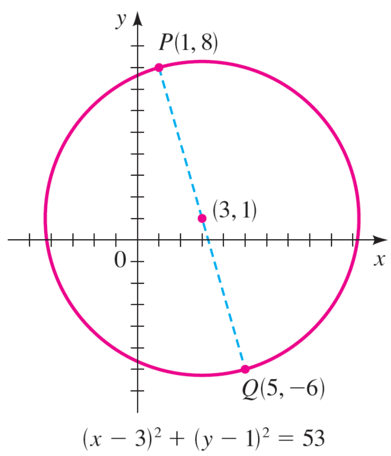
The radius r is the distance from P to the center, so by the Distance Formula

$$r^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 = (3-1)^2 + (1-8)^2 = 2^2 + (-7)^2 = 4 + 49 = 53$$

Therefore, the equation of the circle is

$$(x-3)^2 + (y-1)^2 = 53$$





EXAMPLE: Show that the equation $x^2 + y^2 + 2x - 6y + 7 = 0$ represents a circle, and find the center and radius of the circle.

Solution: We have

$$x^2 + y^2 + 2x - 6y + 7 = 0$$

$$x^2 + 2x + y^2 - 6y = -7$$

$$x^2 + 2x \cdot 1 + y^2 - 2y \cdot 3 = -7$$

$$x^2 + 2x \cdot 1 + 1^2 + y^2 - 2y \cdot 3 + 3^2 = -7 + 1^2 + 3^2$$

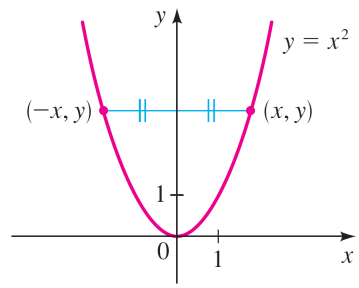
$$(x+1)^2 + (y-3)^2 = 3$$

$$(x-(-1))^2 + (y-3)^2 = (\sqrt{3})^2$$

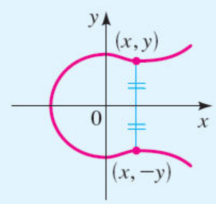
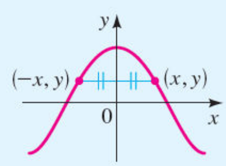
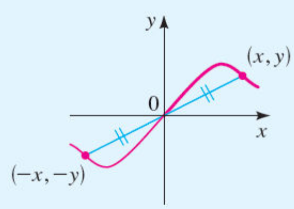
Comparing this equation with the standard equation of a circle, we see that $h = -1$, $k = 3$, and $r = \sqrt{3}$, so the given equation represents a circle with center $(-1, 3)$ and radius $\sqrt{3}$.

Symmetry

The Figure on the right shows the graph of $y = x^2$. Notice that the part of the graph to the left of the y-axis is the mirror image of the part to the right of the y-axis. The reason is that if the point (x, y) is on the graph, then so is $(-x, y)$, and these points are reflections of each other about the y-axis. In this situation we say the graph is **symmetric with respect to the y-axis**. Similarly, we say a graph is **symmetric with respect to the x-axis** if whenever the point (x, y) is on the graph, then so is $(x, -y)$. A graph is **symmetric with respect to the origin** if whenever (x, y) is on the graph, so is $(-x, -y)$.



Definition of Symmetry

Type of symmetry	How to test for symmetry	What the graph looks like (figures in this section)	Geometric meaning
Symmetry with respect to the x-axis	The equation is unchanged when y is replaced by $-y$		Graph is unchanged when reflected in the x-axis
Symmetry with respect to the y-axis	The equation is unchanged when x is replaced by $-x$		Graph is unchanged when reflected in the y-axis
Symmetry with respect to the origin	The equation is unchanged when x is replaced by $-x$ and y by $-y$		Graph is unchanged when rotated 180° about the origin

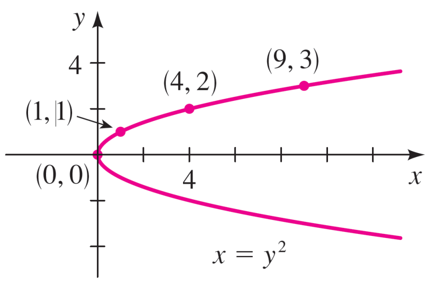
EXAMPLE: Test the equation $x = y^2$ for symmetry and sketch the graph.

EXAMPLE: Test the equation $x = y^2$ for symmetry and sketch the graph.

Solution: If y is replaced by $-y$ in the equation $x = y^2$, we get

$$x = (-y)^2$$
$$x = y^2$$

and so the equation is unchanged. Therefore, the graph is symmetric about the x-axis. But changing x to $-x$ gives the equation $-x = y^2$, which is not the same as the original equation, so the graph is not symmetric about the y-axis. We use the symmetry about the x-axis to sketch the graph by first plotting points just for $y > 0$ and then reflecting the graph in the x-axis, as shown in the Figure below.



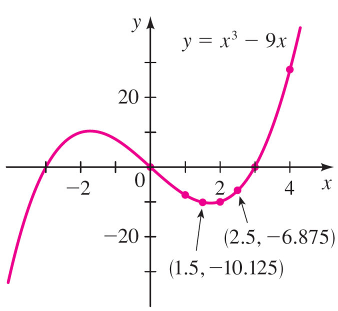
y	$x = y^2$	(x, y)
0	0	(0, 0)
1	1	(1, 1)
2	4	(4, 2)
3	9	(9, 3)

EXAMPLE: Test the equation $y = x^3 - 9x$ for symmetry and sketch the graph.

Solution: If we replace x by $-x$ and y by $-y$ in the equation, we get

$$-y = (-x)^3 - 9(-x)$$
$$-y = -x^3 + 9x$$
$$y = x^3 - 9x$$

and so the equation is unchanged. This means that the graph is symmetric with respect to the origin. We sketch it by first plotting points for $x > 0$ and then using symmetry about the origin.



x	$y = x^3 - 9x$	(x, y)
0	0	(0, 0)
1	−8	(1, −8)
1.5	−10.125	(1.5, −10.125)
2	−10	(2, −10)
2.5	−6.875	(2.5, −6.875)
3	0	(3, 0)
4	28	(4, 28)

Appendix

EXAMPLE: Show that the equation $x^2+y^2+4y+3=0$ represents a circle, and find the center and radius of the circle.

Solution: We have

$$x^2+y^2+4y+3=0$$
$$x^2+y^2+4y=-3$$
$$x^2+y^2+2y\cdot 2=-3$$
$$x^2+y^2+2y\cdot 2+2^2=-3+2^2$$
$$x^2+(y+2)^2=1$$
$$(x-0)^2+(y-(-2))^2=1^2$$

Comparing this equation with the standard equation of a circle, we see that $h=0$, $k=-2$, and $r=1$, so the given equation represents a circle with center $(0,-2)$ and radius 1.

EXAMPLE: Show that the equation $x^2+y^2-6x+2y+8=0$ represents a circle, and find the center and radius of the circle.

Solution: We have

$$x^2+y^2-6x+2y+8=0$$
$$x^2-6x+y^2+2y=-8$$
$$x^2-2x\cdot 3+y^2+2y\cdot 1=-8$$
$$x^2-2x\cdot 3+3^2+y^2+2y\cdot 1+1^2=-8+3^2+1^2$$
$$(x-3)^2+(y+1)^2=2$$
$$(x-3)^2+(y-(-1))^2=(\sqrt{2})^2$$

Comparing this equation with the standard equation of a circle, we see that $h=3$, $k=-1$, and $r=\sqrt{2}$, so the given equation represents a circle with center $(3,-1)$ and radius $\sqrt{2}$.

EXAMPLE: Show that the equation $x^2+y^2+8x-10y+34=0$ represents a circle, and find the center and radius of the circle.

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Solution: We have

$$\begin{aligned} x^2+y^2+8x-10y+34&=0\\ x^2+8x+y^2-10y&=-34\\ x^2+2x\cdot 4+y^2-2y\cdot 5&=-34\\ x^2+2x\cdot 4+4^2+y^2-2y\cdot 5+5^2&=-34+4^2+5^2\\ (x+4)^2+(y-5)^2&=7\\ (x-(-4))^2+(y-5)^2&=(\sqrt{7})^2 \end{aligned}$$

Comparing this equation with the standard equation of a circle, we see that $h=-4,\ k=5$, and $r=\sqrt{7}$, so the given equation represents a circle with center $(-4,5)$ and radius $\sqrt{7}$.

EXAMPLE: Show that the equation $x^2+y^2+x+y-1=0$ represents a circle, and find the center and radius of the circle.

Solution: We have

$$\begin{aligned} x^2+y^2+x+y-1&=0\\ x^2+x+y^2+y&=1\\ x^2+2x\cdot\frac{1}{2}+y^2+2y\cdot\frac{1}{2}&=1\\ x^2+2x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2+y^2+2y\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2&=1+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2\\ \left(x+\frac{1}{2}\right)^2+\left(y+\frac{1}{2}\right)^2&=\frac{3}{2}\\ \left(x-\left(-\frac{1}{2}\right)\right)^2+\left(y-\left(-\frac{1}{2}\right)\right)^2&=\left(\sqrt{\frac{3}{2}}\right)^2 \end{aligned}$$

Comparing this equation with the standard equation of a circle, we see that $h=-1/2,\ k=-1/2$, and $r=\sqrt{3/2}$, so the given equation represents a circle with center $(-1/2,-1/2)$ and radius $\sqrt{3/2}$.

EXAMPLE: Show that the equation $x^2+y^2-x+\frac{6}{5}y-\frac{39}{100}=0$ represents a circle, and find the center and radius of the circle.

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Solution: We have

$$x^2+y^2-x+\frac{6}{5}y-\frac{39}{100}=0$$

$$x^2-x+y^2+\frac{6}{5}y=\frac{39}{100}$$

$$x^2-2x\cdot\frac{1}{2}+y^2+2y\cdot\frac{3}{5}=\frac{39}{100}$$

$$x^2-2x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2+y^2+2y\cdot\frac{3}{5}+\left(\frac{3}{5}\right)^2=\underbrace{\frac{39}{100}+\left(\frac{1}{2}\right)^2+\left(\frac{3}{5}\right)^2}_{1}$$

$$\left(x-\frac{1}{2}\right)^2+\left(y+\frac{3}{5}\right)^2=1$$

$$\left(x-\frac{1}{2}\right)^2+\left(y-\left(-\frac{3}{5}\right)\right)^2=1^2$$

Comparing this equation with the standard equation of a circle, we see that $h=1/2,\ k=-3/5$, and $r=1$, so the given equation represents a circle with center $(1/2,-3/5)$ and radius 1.