Section 1.7 Inequalities

Rules for Inequalities

Rule	Description
1. $A \leq B \quad \Leftrightarrow \quad A + C \leq B + C$	**Adding** the same quantity to each side of an inequality gives an equivalent inequality.
2. $A \leq B \quad \Leftrightarrow \quad A - C \leq B - C$	**Subtracting** the same quantity from each side of an inequality gives an equivalent inequality.
3. If $C > 0$, then $A \leq B \quad \Leftrightarrow \quad CA \leq CB$	**Multiplying** each side of an inequality by the same *positive* quantity gives an equivalent inequality.
4. If $C < 0$, then $A \leq B \quad \Leftrightarrow \quad CA \geq CB$	**Multiplying** each side of an inequality by the same *negative* quantity *reverses the direction* of the inequality.
5. If $A > 0$ and $B > 0$, then $A \leq B \quad \Leftrightarrow \quad \frac{1}{A} \geq \frac{1}{B}$	**Taking reciprocals** of each side of an inequality involving *positive* quantities *reverses the direction* of the inequality.
6. If $A \leq B$ and $C \leq D$, then $A + C \leq B + D$	Inequalities can be added.

Linear Inequalities

An inequality is **linear** if each term is constant or a multiple of the variable.

EXAMPLE: Solve the inequality $3x < 9x + 4$ and sketch the solution set.

Solution: We have

$$\begin{aligned} 3x &< 9x + 4 \\ 3x - 9x &< 9x + 4 - 9x \\ -6x &< 4 \\ \frac{-6x}{-6} &> \frac{4}{-6} \\ x &> -\frac{2}{3} \end{aligned}$$

The solution set consists of all numbers that are greater than $-\frac{2}{3}$. In other words the solution of the inequality is the interval $\left(-\frac{2}{3}, \infty\right)$.

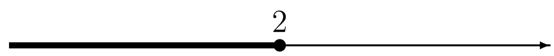
EXAMPLE: Solve the inequality $-5 \geq 6(x - 4) + 7$ and sketch the solution set.

EXAMPLE: Solve the inequality $-5 \geq 6(x-4)+7$ and sketch the solution set.

Solution: We have

$$
\begin{aligned}
-5 &\geq 6(x-4)+7 \\
-5-7 &\geq 6(x-4) \\
-12 &\geq 6x-24 \\
-12+24 &\geq 6x \\
\frac{12}{6} &\geq \frac{6x}{6} \\
2 &\geq x
\end{aligned}
$$

The solution set consists of all numbers that are less than or equal to 2. In other words the solution of the inequality is the interval $(-\infty, 2]$.

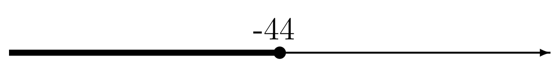


EXAMPLE: Solve the inequality $\frac{1}{2}(x-6)-\frac{4}{3}(x+2) \geq -\frac{3}{4}x-2$ and sketch the solution set.

Solution: We have

$$
\begin{aligned}
\frac{1}{2}(x-6)-\frac{4}{3}(x+2) &\geq -\frac{3}{4}x-2 \\
12\cdot\left(\frac{1}{2}(x-6)-\frac{4}{3}(x+2)\right) &\geq 12\cdot\left(-\frac{3}{4}x-2\right) \\
12\cdot\frac{1}{2}(x-6)-12\cdot\frac{4}{3}(x+2) &\geq 12\cdot\left(-\frac{3}{4}x\right)-12\cdot 2 \\
6(x-6)-16(x+2) &\geq -9x-24 \\
6x-36-16x-32 &\geq -9x-24 \\
6x-16x+9x &\geq -24+36+32 \\
-x &\geq 44 \\
(-1)(-x) &\leq (-1)44 \\
x &\leq -44
\end{aligned}
$$

The solution set consists of all numbers that are less than or equal to -44. In other words the solution of the inequality is the interval $(-\infty, -44]$.



EXAMPLE: Solve the compound inequality $4 \leq 3x - 2 < 13$ and sketch the solution set.

Solution: We have

$$4 \leq 3x - 2 < 13$$

$$4 + 2 \leq 3x - 2 + 2 < 13 + 2$$

$$6 \leq 3x < 15$$

$$\frac{6}{3} \leq \frac{3x}{3} < \frac{15}{3}$$

$$2 \leq x < 5$$

Therefore, the solution set is $[2, 5)$.

EXAMPLE: Solve the compound inequality $-1 < \frac{5 - 3x}{4} \leq \frac{17}{2}$ and sketch the solution set.

Solution: We have

$$-1 < \frac{5 - 3x}{4} \leq \frac{17}{2}$$

$$4 \cdot (-1) < 4 \cdot \frac{5 - 3x}{4} \leq 4 \cdot \frac{17}{2}$$

$$-4 < 5 - 3x \leq 34$$

$$-4 - 5 < 5 - 3x - 5 \leq 34 - 5$$

$$-9 < -3x \leq 29$$

$$\frac{-9}{-3} > \frac{-3x}{-3} \geq \frac{29}{-3}$$

$$3 > x \geq -\frac{29}{3}$$

$$-\frac{29}{3} \leq x < 3$$

Therefore, the solution set is $\left[-\frac{29}{3}, 3\right)$.

Nonlinear Inequalities

Guidelines for Solving Nonlinear Inequalities

1. **Move All Terms to One Side.** If necessary, rewrite the inequality so that all nonzero terms appear on one side of the inequality sign. If the nonzero side of the inequality involves quotients, bring them to a common denominator.
2. **Factor.** Factor the nonzero side of the inequality.
3. **Find the Intervals.** Determine the values for which each factor is zero. These numbers will divide the real line into intervals. List the intervals determined by these numbers.
4. **Make a Table or Diagram.** Use test values to make a table or diagram of the signs of each factor on each interval. In the last row of the table determine the sign of the product (or quotient) of these factors.
5. **Solve.** Determine the solution of the inequality from the last row of the sign table. Be sure to check whether the inequality is satisfied by some or all of the endpoints of the intervals (this may happen if the inequality involves $\le$ or $\ge$).

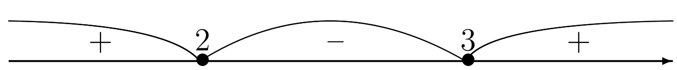
EXAMPLE: Solve the inequality $x^2 \le 5x - 6$ and sketch the solution set.

Solution: The corresponding equation $x^2 - 5x + 6 = (x-2)(x-3) = 0$ has the solutions 2 and 3. As shown in the Figure below, the numbers 2 and 3 divide the real line into three intervals: $(-\infty, 2)$, $(2, 3)$, and $(3, \infty)$.

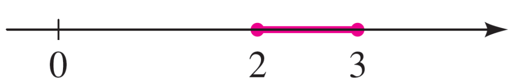
On each of these intervals we determine the signs of the factors using **test values**. We choose a number inside each interval and check the sign of the factors $x-2$ and $x-3$ at the value selected. For instance, if we use the test value $x = 1$ from the interval $(-\infty, 2)$ shown in the Figure above, then substitution in the factors $x-2$ and $x-3$ gives

$$x - 2 = 1 - 2 = -1, \qquad x - 3 = 1 - 3 = -2$$

Both factors are negative on this interval, therefore $x^2 - 5x + 6 = (x-2)(x-3)$ is positive on $(-\infty, 2)$. Similarly, using the test values $x = 2\frac{1}{2}$ and $x = 4$ from the intervals $(2, 3)$ and $(3, \infty)$, respectively, we get:

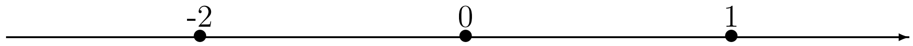


Thus, the solution of the inequality $x^2 \le 5x - 6$ is $\{x \mid 2 \le x \le 3\} = [2, 3]$.



EXAMPLE: Solve the inequality $x(x-1)(x+2) > 0$ and sketch the solution set.

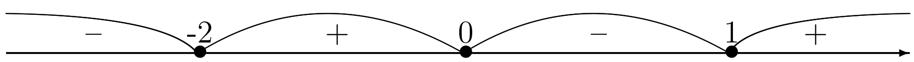
Solution: The corresponding equation $x(x-1)(x+2) = 0$ has the solutions $0, 1$, and -2. As shown in the Figure below, the numbers $0, 1$, and -2 divide the real line into four intervals: $(-\infty, -2)$, $(-2, 0)$, $(0, 1)$, and $(1, \infty)$.



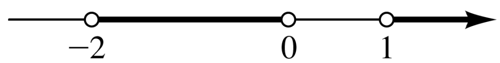
On each of these intervals we determine the signs of the factors using **test values**. We choose a number inside each interval and check the sign of the factors x, $x-1$, and $x+2$ at the value selected. For instance, if we use the test value $x = -3$ from the interval $(-\infty, -2)$ shown in the Figure above, then substitution in the factors x, $x-1$, and $x+2$ gives

$$x = -3, \qquad x - 1 = -3 - 1 = -4, \qquad x + 2 = -3 + 2 = -1$$

All three factors are negative on this interval, therefore $x(x-1)(x+2)$ is negative on $(-\infty, -2)$. Similarly, using the test values $x = -1$, $x = 1/2$ and $x = 2$ from the intervals $(-2, 0)$, $(0, 1)$ and $(1, \infty)$, respectively, we get:

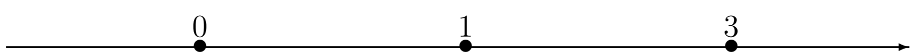


Thus, the solution of the inequality $x(x-1)(x+2) > 0$ is $(-2, 0) \cup (1, \infty)$.



EXAMPLE: Solve the inequality $x(x-1)^2(x-3) < 0$.

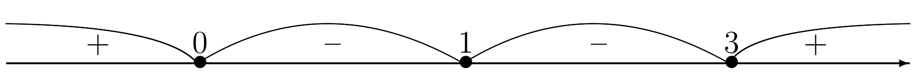
Solution: The corresponding equation $x(x-1)^2(x-3) = 0$ has the solutions $0, 1$, and 3. As shown in the Figure below, the numbers $0, 1$, and 3 divide the real line into four intervals: $(-\infty, 0)$, $(0, 1)$, $(1, 3)$, and $(3, \infty)$.



On each of these intervals we determine the signs of the factors using **test values**. We choose a number inside each interval and check the sign of the factors x and $x-3$ at the value selected. For instance, if we use the test value $x = -1$ from the interval $(-\infty, 0)$ shown in the Figure above, then substitution in the factors x and $x-3$ gives

$$x = -1, \qquad x - 3 = -1 - 3 = -4$$

Both factors are negative on this interval, $(x-1)^2$ is always nonnegative, therefore $x(x-1)^2(x-3)$ is positive on $(-\infty, 0)$. Similarly, using the test values $x = 1/2$, $x = 2$, and $x = 4$ from the intervals $(0, 1)$, $(1, 3)$, and $(3, \infty)$, respectively, we get:



Thus, the solution of the inequality $x(x-1)^2(x-3) < 0$ is $(0, 1) \cup (1, 3)$.

REMARK: The solution of the inequality $x(x-1)^2(x-3) \le 0$ is $[0, 3]$.

EXAMPLE: Solve the following inequalities:

(a) $(x+3)(x+1) \geq 8$ (b) $(x+2)(x+4) \leq -1$ (c) $(x+2)(x+4) \geq -1$

Solution:

(a) We have

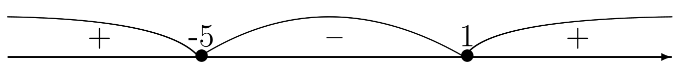
$$
\begin{aligned}
(x+3)(x+1) &\geq 8 \\
x^2+4x+3 &\geq 8 \\
x^2+4x+3-8 &\geq 0 \\
x^2+4x-5 &\geq 0 \\
(x+5)(x-1) &\geq 0
\end{aligned}
$$

The corresponding equation $(x+5)(x-1)=0$ has the solutions -5 and 1. As shown in the Figure below, the numbers -5 and 1 divide the real line into three intervals: $(-\infty,-5)$, $(-5,1)$, and $(1,\infty)$.

On each of these intervals we determine the signs of the factors using **test values**. We choose a number inside each interval and check the sign of the factors $x+5$ and $x-1$ at the value selected. For instance, if we use the test value $x=-6$ from the interval $(-\infty,-5)$ shown in the Figure above, then substitution in the factors $x+5$ and $x-1$ gives

$$x+5=-6+5=-1, \qquad x-1=-6-1=-7$$

Both factors are negative on this interval, therefore $(x+5)(x-1)$ is positive on $(-\infty,5)$. Similarly, using the test values $x=0$ and $x=2$ from the intervals $(-5,1)$ and $(1,\infty)$, respectively, we get:



Thus, the solution of the inequality $(x+3)(x+1) \geq 8$ is $\{x \mid x \leq -5 \text{ or } x \geq 1\} = (-\infty,-5] \cup [1,\infty)$.

(b, c) We have

$$
\begin{aligned}
(x+2)(x+4) &\leq -1 \\
x^2+6x+8 &\leq -1 \\
x^2+6x+8+1 &\leq 0 \\
\underbrace{x^2+6x+9}_{x^2+2\cdot x\cdot 3+3^2} &\leq 0 \\
(x+3)^2 &\leq 0
\end{aligned}
$$

Note that $(x+3)^2$ is either positive or zero. Therefore, the only solution of the inequality $(x+2)(x+4) \leq -1$ is $x=-3$. In a similar way one can show that the solution of the inequality $(x+2)(x+4) \geq -1$ is all real numbers.

EXAMPLE: Solve the inequality $\dfrac{1+x}{1-x} \geq 1$ and sketch the solution set.

Solution: We have

$$\frac{1+x}{1-x} \geq 1$$

$$\frac{1+x}{1-x} - 1 \geq 0$$

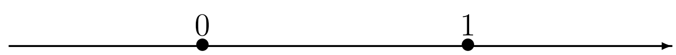
$$\frac{1+x}{1-x} - \frac{1-x}{1-x} \geq 0$$

$$\frac{1+x-(1-x)}{1-x} \geq 0$$

$$\frac{1+x-1+x}{1-x} \geq 0$$

$$\frac{2x}{1-x} \geq 0$$

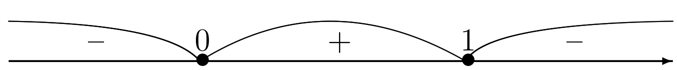
As shown in the Figure below, the numbers 0 (at which the numerator of $\dfrac{2x}{1-x}$ is 0) and 1 (at which the denominator of $\dfrac{2x}{1-x}$ is 0) divide the real line into three intervals: $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$.



On each of these intervals we determine the sign of $\dfrac{2x}{1-x}$ using **test values**. We choose a number inside each interval and check the sign of $\dfrac{2x}{1-x}$ at the value selected. For instance, if we use the test value $x = -1$ from the interval $(-\infty, 0)$ shown in the Figure above, then substitution in $\dfrac{2x}{1-x}$ gives

$$\frac{2(-1)}{1-(-1)} = \frac{-2}{2} = -1 < 0$$

Similarly, using the test values $x = \dfrac{1}{2}$ and $x = 2$ from the intervals $(0, 1)$, and $(1, \infty)$, respectively, we get:



Thus, the solution of the inequality $\dfrac{1+x}{1-x} \geq 1$ is $[0, 1)$.

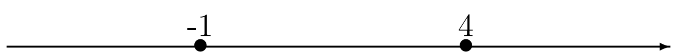
EXAMPLE: Solve the inequality $\dfrac{2x-3}{x+1} \leq 1$ and sketch the solution set.

EXAMPLE: Solve the inequality $\dfrac{2x-3}{x+1} \leq 1$ and sketch the solution set.

Solution: We have

$$\begin{aligned}
\frac{2x-3}{x+1} &\leq 1\\
\frac{2x-3}{x+1} - 1 &\leq 0\\
\frac{2x-3}{x+1} - \frac{x+1}{x+1} &\leq 0\\
\frac{2x-3-(x+1)}{x+1} &\leq 0\\
\frac{2x-3-x-1}{x+1} &\leq 0\\
\frac{x-4}{x+1} &\leq 0
\end{aligned}$$

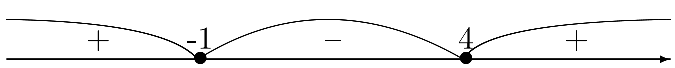
As shown in the Figure below, the numbers 4 (at which the numerator of $\dfrac{x-4}{x+1}$ is 0) and -1 (at which the denominator of $\dfrac{x-4}{x+1}$ is 0) divide the real line into three intervals: $(-\infty,-1)$, $(-1,4)$, and $(4,\infty)$.



On each of these intervals we determine the sign of $\dfrac{x-4}{x+1}$ using **test values**. We choose a number inside each interval and check the sign of $\dfrac{x-4}{x+1}$ at the value selected. For instance, if we use the test value $x=-2$ from the interval $(-\infty,-1)$ shown in the Figure above, then substitution in $\dfrac{x-4}{x+1}$ gives

$$\frac{-2-4}{-2+1} = \frac{-6}{-1} = 6 > 0$$

Similarly, using the test values $x=0$ and $x=5$ from the intervals $(-1,4)$, and $(4,\infty)$, respectively, we get:



Thus, the solution of the inequality $\dfrac{2x-3}{x+1} \leq 1$ is $(-1,4]$.

EXAMPLE: Solve the inequality $\dfrac{7-5x}{8x+9} < \dfrac{2}{3}$ and sketch the solution set.

EXAMPLE: Solve the inequality $\dfrac{7-5x}{8x+9}<\dfrac{2}{3}$ and sketch the solution set.

Solution: We have

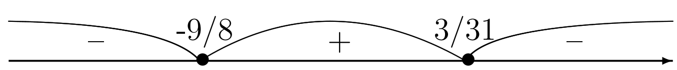
$$\begin{aligned}\frac{7-5x}{8x+9}&<\frac{2}{3}\\ \frac{7-5x}{8x+9}-\frac{2}{3}&<0\\ \frac{3(7-5x)}{3(8x+9)}-\frac{2(8x+9)}{3(8x+9)}&<0\\ \frac{3(7-5x)-2(8x+9)}{3(8x+9)}&<0\\ \frac{21-15x-16x-18}{3(8x+9)}&<0\\ \frac{3-31x}{3(8x+9)}&<0\end{aligned}$$

As shown in the Figure below, the numbers $3/31$ (at which the numerator of $\dfrac{3-31x}{3(8x+9)}$ is 0) and $-9/8$ (at which the denominator of $\dfrac{3-31x}{3(8x+9)}$ is 0) divide the real line into three intervals: $(-\infty,-9/8)$, $(-9/8,3/31)$, and $(3/31,\infty)$.

On each of these intervals we determine the sign of $\dfrac{3-31x}{3(8x+9)}$ using **test values**. We choose a number inside each interval and check the sign of $\dfrac{3-31x}{3(8x+9)}$ at the value selected. For instance, if we use the test value $x=-2$ from the interval $(-\infty,-9/8)$ shown in the Figure above, then substitution in $\dfrac{3-31x}{3(8x+9)}$ gives

$$\frac{3-31x}{3(8x+9)}=\frac{3-31(-2)}{3(8(-2)+9)}=\frac{3+62}{3(-16+9)}=\frac{65}{3(-7)}<0$$

Similarly, using the test values $x=0$ and $x=1$ from the intervals $(-9/8,3/31)$, and $(3/31,\infty)$, respectively, we get:



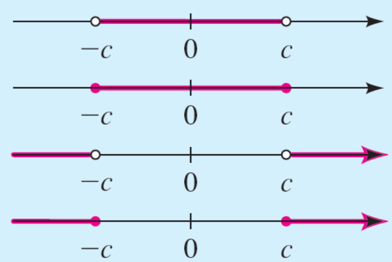
Thus, the solution of the inequality $\dfrac{7-5x}{8x+9}<\dfrac{2}{3}$ is $\left(-\infty,-\dfrac{9}{8}\right)\cup\left(\dfrac{3}{31},\infty\right)$.

Absolute Value Inequalities

Properties of Absolute Value Inequalities

Inequality	Equivalent form
1. $\lvert x \rvert < c$	$-c < x < c$
2. $\lvert x \rvert \leq c$	$-c \leq x \leq c$
3. $\lvert x \rvert > c$	$x < -c$ or $c < x$
4. $\lvert x \rvert \geq c$	$x \leq -c$ or $c \leq x$

Graph



EXAMPLE: Solve the inequality $|x-5|<2$ and sketch the solution set.

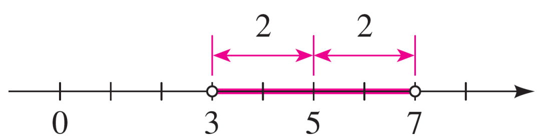
Solution: The inequality $|x-5|<2$ is equivalent to

$$-2 < x-5 < 2$$

$$-2+5 < x-5+5 < 2+5$$

$$3 < x < 7$$

The solution set is the open interval $(3,7)$.



EXAMPLE: Solve the inequality $|x-5|<-2$.

Solution: The inequality $|x-5|<-2$ has no solutions, since $|\cdot|$ is always nonnegative. So, the solution set is the empty set.

EXAMPLE: Solve the inequality $|7-3x|<1$.

EXAMPLE: Solve the inequality $|7-3x|<1$.

Solution: The inequality $|7-3x|<1$ is equivalent to

$$\begin{gathered}
-1<7-3x<1\\
-1-7<7-3x-7<1-7\\
-8<-3x<-6\\
\frac{-8}{-3}>\frac{-3x}{-3}>\frac{-6}{-3}\\
2<x<\frac{8}{3}
\end{gathered}$$

The solution set is the open interval $\left(2,\dfrac{8}{3}\right)$.

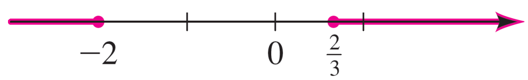
EXAMPLE: Solve the inequality $|3x+2|\geq 4$ and sketch the solution set.

Solution: The inequality $|3x+2|\geq 4$ is equivalent to

$$\begin{aligned}
3x+2&\geq 4 &\qquad\text{or}\qquad 3x+2&\leq -4\\
3x&\geq 2 & 3x&\leq -6\\
x&\geq \frac{2}{3} & x&\leq -2
\end{aligned}$$

The solution set is

$$\left\{x \mid x\leq -2 \text{ or } x\geq \frac{2}{3}\right\}=(-\infty,-2]\cup\left[\frac{2}{3},\infty\right)$$



EXAMPLE: Solve the inequality $4|5x+9|-7\leq 2$.

Solution: We have

$$4|5x+9|-7\leq 2 \quad\Longleftrightarrow\quad 4|5x+9|\leq \underbrace{2+7}_{9} \quad\Longleftrightarrow\quad |5x+9|\leq\frac{9}{4}$$

The inequality $|5x+9|\leq\dfrac{9}{4}$ is equivalent to

$$\begin{gathered}
-\frac{9}{4}\leq 5x+9\leq\frac{9}{4}\\
4\cdot\left(-\frac{9}{4}\right)\leq 4\cdot(5x+9)\leq 4\cdot\frac{9}{4}\\
-9\leq 20x+36\leq 9\\
-9-36\leq 20x\leq 9-36\\
-45\leq 20x\leq -27\\
-\frac{9}{4}=-\frac{45}{20}\leq x\leq -\frac{27}{20}
\end{gathered}$$

The solution set is $\left\{x \mid -\dfrac{9}{4}\leq x\leq -\dfrac{27}{20}\right\}=\left[-\dfrac{9}{4},-\dfrac{27}{20}\right]$.