Section 1.6 Modeling with Equations

Guidelines for Modeling with Equations

1. **Identify the Variable.** Identify the quantity that the problem asks you to find. This quantity can usually be determined by a careful reading of the question posed at the end of the problem. Then **introduce notation** for the variable (call it x or some other letter).

2. **Express All Unknown Quantities in Terms of the Variable.** Read each sentence in the problem again, and express all the quantities mentioned in the problem in terms of the variable you defined in Step 1. To organize this information, it is sometimes helpful to **draw a diagram** or **make a table**.

3. **Set Up the Model.** Find the crucial fact in the problem that gives a relationship between the expressions you listed in Step 2. **Set up an equation** (or **model**) that expresses this relationship.

4. **Solve the Equation and Check Your Answer.** Solve the equation, check your answer, and express it as a sentence that answers the question posed in the problem.

EXAMPLE: A car rental company charges \$30 a day and 15c a mile for renting a car. Helen rents a car for two days and her bill comes to \$108. How many miles did she drive?

Solution: We are asked to find the number of miles Helen has driven. So we let

$$x = \text{ number of miles driven}$$

Then we translate all the information given in the problem into the language of algebra.

In Words	In Algebra
Number of miles driven	x
Mileage cost (at \$0.15 per mile)	$0.15x$
Daily cost (at \$30 per day)	$2(30)$

Now we set up the model.

$$\boxed{\text{milage cost}} + \boxed{\text{daily cost}} = \boxed{\text{total cost}}$$

$$0.15x + 2(30) = 108$$

$$0.15x = 108 - 2(30)$$

$$0.15x = 48$$

$$x = \frac{48}{0.15} = \left\{ \frac{48 \cdot 100}{0.15 \cdot 100} = \frac{48 \cdot 100}{15} = \frac{(3 \cdot 16) \cdot (5 \cdot 20)}{3 \cdot 5} = 16 \cdot 20 \right\} = 320$$

Helen drove her rental car 320 miles.

EXAMPLE: Mary inherits \$100,000 and invests it in two certificates of deposit. One certificate pays 6% and the other pays $4\frac{1}{2}\%$ simple interest annually. If Mary's total interest is \$5025 per year, how much money is invested at each rate?

Solution: The problem asks for the amount she has invested at each rate. So we let

$$x = \text{ the amount invested at } 6\%$$

Since Mary's total inheritance is \$100,000, it follows that she invested $100,000 - x$ at $4\frac{1}{2}\%$. We translate all the information given into the language of algebra.

In Words	In Algebra
Amount invested at 6%	x
Amount invested at $4\frac{1}{2}\%$	$100{,}000 - x$
Interest earned at 6%	$0.06x$
Interest earned at $4\frac{1}{2}\%$	$0.045(100{,}000 - x)$

We use the fact that Mary's total interest is \$5025 to set up the model.

$$\boxed{\text{interest at } 6\%} + \boxed{\text{interest at } 4\tfrac{1}{2}\%} = \boxed{\text{total interest}}$$

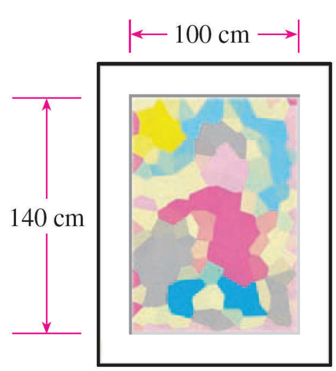
$$\begin{aligned} 0.06x + 0.045(100,000 - x) &= 5025 \\ 0.06x + 4500 - 0.045x &= 5025 \\ 0.015x + 4500 &= 5025 \\ 0.015x &= 525 \end{aligned}$$

therefore

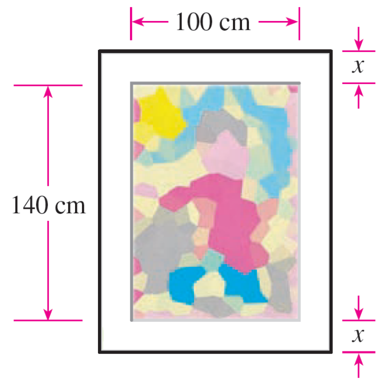
$$x = \frac{525}{0.015} = \left\{ \frac{525 \cdot 1000}{0.015 \cdot 1000} = \frac{(5 \cdot 105) \cdot 1000}{15} = \frac{(5 \cdot 3 \cdot 35) \cdot 1000}{3 \cdot 5} = 35 \cdot 1000 \right\} = 35,000$$

So Mary has invested \$35,000 at 6% and the remaining \$65,000 at $4\frac{1}{2}\%$.

EXAMPLE: A poster has a rectangular printed area 100 cm by 140 cm, and a blank strip of uniform width around the four edges. The perimeter of the poster is $1\frac{1}{2}$ times the perimeter of the printed area. What is the width of the blank strip, and what are the dimensions of the poster?



EXAMPLE: A poster has a rectangular printed area 100 cm by 140 cm, and a blank strip of uniform width around the four edges. The perimeter of the poster is $1\frac{1}{2}$ times the perimeter of the printed area. What is the width of the blank strip, and what are the dimensions of the poster?



Solution: We are asked to find the width of the blank strip. So we let

$$x = \text{ the width of the blank strip}$$

Then we translate the information in the Figure above into the language of algebra:

In Words	In Algebra
Width of blank strip	x
Perimeter of printed area	$2(100) + 2(140) = 480$
Width of poster	$100 + 2x$
Length of poster	$140 + 2x$
Perimeter, of poster	$2(100 + 2x) + 2(140 + 2x)$

Now we use the fact that the perimeter of the poster is $1\frac{1}{2}$ times the perimeter of the printed area.

$$\boxed{\text{perimeter of poster}} = \frac{3}{2} \cdot \boxed{\text{perimeter of printed area}}$$

$$2(100 + 2x) + 2(140 + 2x) = \frac{3}{2} \cdot 480$$

$$200 + 4x + 280 + 4x = 3 \cdot 240$$

$$480 + 8x = 720$$

$$8x = 240$$

$$x = 30$$

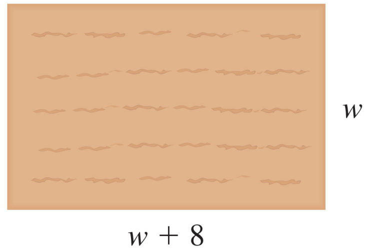
The blank strip is 30 cm wide, so the dimensions of the poster are

$$100 + 30 + 30 = 160 \text{ cm wide}$$

by

$$140 + 30 + 30 = 200 \text{ cm long}$$

EXAMPLE: A rectangular building lot is 8 ft longer than it is wide and has an area of 2900 ft^2. Find the dimensions of the lot.



Solution: We are asked to find the width and length of the lot. So let

$$w = \text{ width of lot}$$

Then we translate the information given in the problem into the language of algebra.

In Words	In Algebra
Width of lot	w
Length of lot	$w + 8$

Now we set up the model.

$$\boxed{\text{width of lot}} \cdot \boxed{\text{length of lot}} = \boxed{\text{area of lot}}$$

$$w(w+8) = 2900$$

$$w^2 + 8w = 2900$$

$$w^2 + 8w - 2900 = 0$$

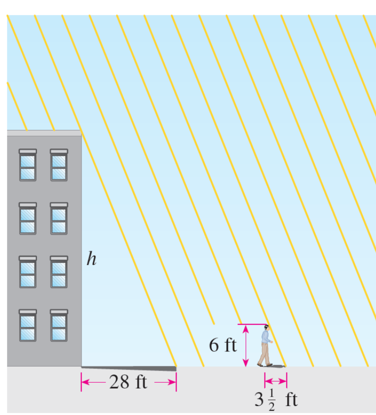
$$(w-50)(w+58) = 0$$

which gives

$$w = 50 \quad \text{or} \quad w = -58$$

Since the width of the lot must be a positive number, we conclude that $w = 50$ ft. The length of the lot is $w + 8 = 50 + 8 = 58$ ft.

EXAMPLE: A man 6 ft tall wishes to find the height of a certain four-story building. He measures its shadow and finds it to be 28 ft long, while his own shadow is $3\frac{1}{2}$ ft long. How tall is the building?



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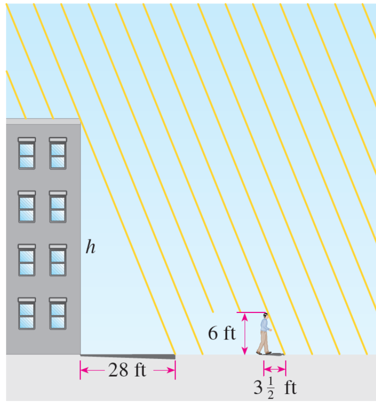

Solution: The problem asks for the height of the building. So let

$$h = \text{ the height of the building}$$

We use the fact that the triangles in the Figure above are similar. Recall that for any pair of similar triangles the ratios of corresponding sides are equal. Now we translate these observations into the language of algebra.

In Words	In Algebra
Height of building	h
Ratio of height to base in large triangle	$\frac{h}{28}$
Ratio of height to base in small triangle	$\frac{6}{3.5}$

Since the large and small triangles are similar, we get the equation

$$\boxed{\text{ratio of height to base in large triangle}} = \boxed{\text{ratio of height to base in small triangle}}$$

$$\frac{h}{28} = \frac{6}{3.5}$$

$$h = \frac{6 \cdot 28}{3.5} = \left\{ \frac{6 \cdot 28 \cdot 10}{3.5 \cdot 10} = \frac{6 \cdot 28 \cdot 10}{35} = \frac{6 \cdot (4 \cdot 7) \cdot (2 \cdot 5)}{5 \cdot 7} = 6 \cdot 4 \cdot 2 \right\} = 48$$

or

$$= \left\{ \frac{6 \cdot 28 \cdot 2}{3.5 \cdot 2} = \frac{6 \cdot 28 \cdot 2}{7} = \frac{6 \cdot (4 \cdot 7) \cdot 2}{7} = 6 \cdot 4 \cdot 2 \right\} = 48$$

The building is 48 ft tall.

EXAMPLE: A manufacturer of soft drinks advertises their orange soda as "naturally flavored," although it contains only 5% orange juice. A new federal regulation stipulates that to be called "natural" a drink must contain at least 10% fruit juice. How much pure orange juice must this manufacturer add to 900 gal of orange soda to conform to the new regulation?

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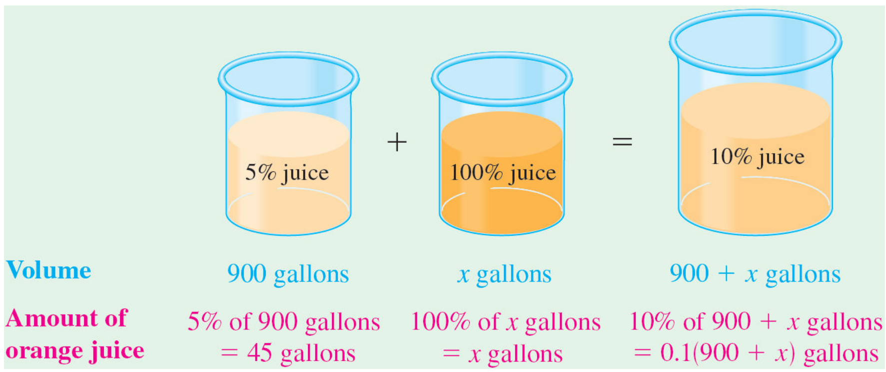
Solution: The problem asks for the amount of pure orange juice to be added. So let

$$x = \text{ the amount (in gallons) of pure orange juice to be added}$$

In any problem of this type — in which two different substances are to be mixed — drawing a diagram helps us organize the given information.



We now translate the information in the figure into the language of algebra.

In Words	In Algebra
Amount of orange juice to be added	x
Amount of the mixture	$900 + x$
Amount of orange juice in the first vat	$0.05(900) = 45$
Amount of orange juice in the second vat	$1 \cdot x = x$
Amount of orange juice in the mix ture	$0.10(900 + x)$

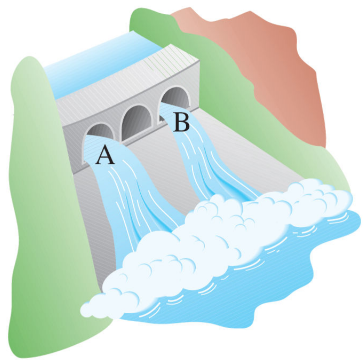
To set up the model, we use the fact that the total amount of orange juice in the mixture is equal to the orange juice in the first two vats.

$$\boxed{\begin{array}{l}\text{amount of orange}\\ \text{juice in first vat}\end{array}} + \boxed{\begin{array}{l}\text{amount of orange}\\ \text{juice in second vat}\end{array}} = \boxed{\begin{array}{l}\text{amount of orange}\\ \text{juice in mixture}\end{array}}$$

$$\begin{aligned} 45 + x &= 0.1(900 + x) \\ 45 + x &= 90 + 0.1x \\ x - 0.1x &= 90 - 45 \\ 0.9x &= 45 \\ x &= \frac{45}{0.9} = \left\{\frac{45 \cdot 10}{0.9 \cdot 10} = \frac{45 \cdot 10}{9} = \frac{(5 \cdot 9) \cdot 10}{9} = 5 \cdot 10\right\} = 50 \end{aligned}$$

The manufacturer should add 50 gal of pure orange juice to the soda.

EXAMPLE: Because of an anticipated heavy rainstorm, the water level in a reservoir must be lowered by 1 ft. Opening spillway A lowers the level by this amount in 4 hours, whereas opening the smaller spillway B does the job in 6 hours. How long will it take to lower the water level by 1 ft if both spillways are opened?



Solution: We are asked to find the time needed to lower the level by 1 ft if both spillways are open. So let

$x =$ the time (in hours) it takes to lower the water level by 1 ft if both spillways are open

Finding an equation relating x to the other quantities in this problem is not easy. Certainly x is not simply $4+6$, because that would mean that together the two spillways require longer to lower the water level than either spillway alone. Instead, *we look at the fraction of the job that can be done in one hour by each spillway.*

In Words	In Algebra
Time it takes to lower level 1 ft with A and B together	x h
Distance A lowers level in 1 h	$\frac{1}{4}$ ft
Distance B lowers level in 1 h	$\frac{1}{6}$ ft
Distance A and B together lower levels in 1 h	$\frac{1}{x}$ ft

Now we set up the model.

$$\boxed{\text{fraction done by A}} + \boxed{\text{fraction done by B}} = \boxed{\text{fraction done by both}}$$

$$\begin{aligned}
\frac{1}{4}+\frac{1}{6} &= \frac{1}{x}\\
12x\cdot\left(\frac{1}{4}+\frac{1}{6}\right) &= 12x\cdot\frac{1}{x}\\
12x\cdot\frac{1}{4}+12x\cdot\frac{1}{6} &= 12x\cdot\frac{1}{x}\\
3x+2x &= 12\\
5x &= 12\\
x &= \frac{12}{5}
\end{aligned}$$

It will take $2\frac{2}{5}$ hours, or 2 h 24 min, to lower the water level by 1 ft if both spillways are open.

EXAMPLE: A jet flew from New York to Los Angeles, a distance of 4200 km. The speed for the return trip was 100 km/h faster than the outbound speed. If the total trip took 13 hours, what was the jet's speed from New York to Los Angeles?

Solution: We are asked for the speed of the jet from New York to Los Angeles. So let

$$s = \text{ speed from New York to Los Angeles}$$

Then

$$s + 100 = \text{ speed from Los Angeles to New York}$$

Now we organize the information in a table. We fill in the "Distance" column first, since we know that the cities are 4200 km apart. Then we fill in the "Speed" column, since we have expressed both speeds (rates) in terms of the variable s. Finally, we calculate the entries for the "Time" column, using

$$\text{time } = \frac{\text{distance}}{\text{rate}}$$

	Distance (km)	Speed (km/h)	Time (h)
N.Y. to L.A.	4200	s	$\frac{4200}{s}$
L.A. to N.Y.	4200	$s + 100$	$\frac{4200}{s + 100}$

The total trip took 13 hours, so we have the model

$$\boxed{\text{time from N.Y. to L.A.}} + \boxed{\text{time from L.A. to N.Y.}} = \boxed{\text{total time}}$$

$$\begin{aligned}
\frac{4200}{s} + \frac{4200}{s+100} &= 13 \\
4200(s+100) + 4200s &= 13s(s+100) \\
8400s + 420,000 &= 13s^2 + 1300s \\
0 &= 13s^2 - 7100s - 420,000
\end{aligned}$$

Although this equation does factor, with numbers this large it is probably quicker to use the quadratic formula and a calculator.

$$s = \frac{7100 \pm \sqrt{(-7100)^2 - 4(13)(-420,000)}}{2(13)} = \frac{7100 \pm 8500}{26}$$

which gives

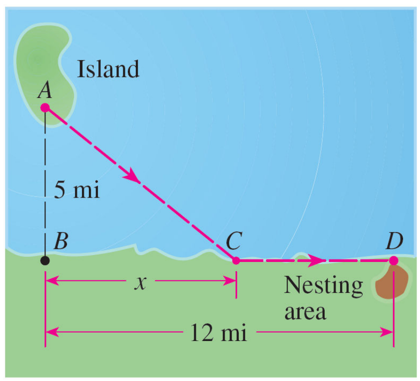
$$s = 600 \quad \text{or} \quad s = -\frac{1400}{26}$$

Since s represents speed, we reject the negative answer and conclude that the jet's speed from New York to Los Angeles was 600 km/h.

EXAMPLE: Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours, because air generally rises over land and falls over water in the daytime, so flying over water requires more energy. A bird is released from point A on an island, 5 mi from B, the nearest point on a straight shoreline. The bird flies to a point C on the shoreline and then flies along the shoreline to its nesting area D, as shown in the Figure below. Suppose the bird has 170 kcal of energy reserves. It uses 10 kcal/mi flying over land and 14 kcal/mi flying over water.

(a) Where should the point C be located so that the bird uses exactly 170 kcal of energy during its flight?

(b) Does the bird have enough energy reserves to fly directly from A to D?



Solution:

(a) We are asked to find the location of C. So let

$$x = \text{ distance from } B \text{ to } C$$

From the figure, and from the fact that

$$\boxed{\text{energy used}} = \boxed{\text{energy per mile}} \times \boxed{\text{miles flown}}$$

we determine the following:

In Words	In Algebra	
Distance from B to C	x	
Distance flown over water (from A to C)	$\sqrt{x^2+25}$	Pythagorean Theorem
Distance flown over land (from C to D)	$12-x$	
Energy used over water	$14\sqrt{x^2+25}$	
Energy used over land	$10(12-x)$	

Now we set up the model.

$$\boxed{\text{total energy used}} = \boxed{\text{energy used over water}} + \boxed{\text{energy used over land}}$$

$$\begin{aligned}
170 &= 14\sqrt{x^2+25} + 10(12-x)\\
170 - 10(12-x) &= 14\sqrt{x^2+25}\\
170 - 120 + 10x &= 14\sqrt{x^2+25}\\
50 + 10x &= 14\sqrt{x^2+25}\\
(50+10x)^2 &= 14^2(x^2+25)\\
2500 + 1000x + 100x^2 &= 196x^2 + 4900\\
0 &= 96x^2 - 1000x + 2400
\end{aligned}$$

This equation could be factored, but because the numbers are so large it is easier to use the quadratic formula and a calculator:

$$x = \frac{1000 \pm \sqrt{(-1000)^2 - 4(96)(2400)}}{2(96)} = \frac{1000 \pm 280}{192}$$

which gives

$$x = 6\frac{2}{3} \quad \text{or} \quad x = 3\frac{3}{4}$$

Point C should be either $6\frac{2}{3}$ mi or $3\frac{3}{4}$mi from B so that the bird uses exactly 170 kcal of energy during its flight.

(b) By the Pythagorean Theorem, the length of the route directly from A to D is $\sqrt{5^2+12^2} = 13$ mi, so the energy the bird requires for that route is $14 \times 13 = 182$ kcal. This is more energy than the bird has available, so it can't use this route.