Section 1.5 Equations

Linear and Rational Equations

Linear Equations

A **linear equation** in one variable is an equation equivalent to one of the form

$$ax + b = 0$$

where a and b are real numbers and x is the variable.

EXAMPLES:

1. Solve the equation $7x - 4 = 3x + 8$.

Solution: We have

$$\begin{aligned} 7x - 4 &= 3x + 8 \\ 7x - 4 + 4 &= 3x + 8 + 4 \\ 7x &= 3x + 12 \\ 7x - 3x &= 3x + 12 - 3x \\ 4x &= 12 \\ \frac{4x}{4} &= \frac{12}{4} \\ x &= 3 \end{aligned}$$

or, in short,

$$\begin{aligned} 7x - 4 &= 3x + 8 \\ 7x - 3x &= 8 + 4 \\ 4x &= 12 \\ x &= \frac{12}{4} = 3 \end{aligned}$$

2. Solve the equation $-5(x - 4) + 2 = 2(x + 7) - 3$.

Solution: We have

$$\begin{aligned} -5(x - 4) + 2 &= 2(x + 7) - 3 \\ -5x + 20 + 2 &= 2x + 14 - 3 \\ -5x + 22 &= 2x + 11 \\ -5x + 22 - 2x &= 2x + 11 - 2x \\ -7x + 22 &= 11 \\ -7x + 22 - 22 &= 11 - 22 \\ -7x &= -11 \\ \frac{-7x}{-7} &= \frac{-11}{-7} \\ x &= \frac{11}{7} \end{aligned}$$

or, in short,

$$\begin{aligned} -5(x - 4) + 2 &= 2(x + 7) - 3 \\ -5x + 20 + 2 &= 2x + 14 - 3 \\ -5x - 2x &= 14 - 3 - 20 - 2 \\ -7x &= -11 \\ x &= \frac{11}{7} \end{aligned}$$

3. Solve the equation $\frac{y+5}{2} - \frac{y-2}{4} = \frac{y+7}{3} + 1$.

Solution: We have

$$\begin{aligned}
\frac{y+5}{2} - \frac{y-2}{4} &= \frac{y+7}{3} + 1 \\
12 \cdot \left(\frac{y+5}{2} - \frac{y-2}{4}\right) &= 12 \cdot \left(\frac{y+7}{3} + 1\right) \\
12 \cdot \frac{y+5}{2} - 12 \cdot \frac{y-2}{4} &= 12 \cdot \frac{y+7}{3} + 12 \cdot 1 \\
6(y+5) - 3(y-2) &= 4(y+7) + 12 \\
6y + 30 - 3y + 6 &= 4y + 28 + 12 \\
3y + 36 &= 4y + 40 \\
3y + 36 - 4y &= 4y + 40 - 4y \\
-y + 36 &= 40 \\
-y + 36 - 36 &= 40 - 36 \\
-y &= 4 \\
-y(-1) &= 4(-1) \\
y &= -4
\end{aligned}$$

In short,

$$\begin{aligned}
\frac{y+5}{2} - \frac{y-2}{4} &= \frac{y+7}{3} + 1 \\
12 \cdot \left(\frac{y+5}{2} - \frac{y-2}{4}\right) &= 12 \cdot \left(\frac{y+7}{3} + 1\right) \\
6(y+5) - 3(y-2) &= 4(y+7) + 12 \\
6y + 30 - 3y + 6 &= 4y + 28 + 12 \\
30 + 6 - 28 - 12 &= 4y - 6y + 3y \\
y &= -4
\end{aligned}$$

4. Solve the equation $\frac{15}{y} = \frac{21}{4y} + 2$.

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Solution: We have

$$\begin{aligned}
\frac{15}{y} &= \frac{21}{4y} + 2 \\
4y \cdot \frac{15}{y} &= 4y \cdot \left(\frac{21}{4y} + 2\right) \\
4y \cdot \frac{15}{y} &= 4y \cdot \frac{21}{4y} + 4y \cdot 2 \\
4 \cdot 15 &= 21 + 8y \\
60 &= 21 + 8y \\
60 - 21 &= 21 + 8y - 21 \\
39 &= 8y \\
\frac{39}{8} &= \frac{8y}{8} \\
y &= \frac{39}{8}
\end{aligned}$$

or, in short,

$$\begin{aligned}
\frac{15}{y} &= \frac{21}{4y} + 2 \\
4y \cdot \frac{15}{y} &= 4y \cdot \left(\frac{21}{4y} + 2\right) \\
60 &= 21 + 8y \\
39 &= 8y \\
y &= \frac{39}{8}
\end{aligned}$$

5. Solve the equation $\dfrac{y}{y+5} = \dfrac{-5}{y+5} + \dfrac{5}{4}$.

Solution: We have

$$\begin{aligned}
\frac{y}{y+5} &= \frac{-5}{y+5} + \frac{5}{4} \\
4(y+5) \cdot \frac{y}{y+5} &= 4(y+5) \cdot \left(\frac{-5}{y+5} + \frac{5}{4}\right) \\
4(y+5) \cdot \frac{y}{y+5} &= 4(y+5) \cdot \frac{-5}{y+5} + 4(y+5) \cdot \frac{5}{4} \\
4y &= 4 \cdot (-5) + (y+5) \cdot 5 \\
4y &= -20 + 5y + 25 \\
4y &= 5 + 5y \\
4y - 5y &= 5 + 5y - 5y \\
-y &= 5 \\
(-1)(-y) &= (-1)5 \\
y &= -5
\end{aligned}$$

or, in short,

$$\begin{aligned}
\frac{y}{y+5} &= \frac{-5}{y+5} + \frac{5}{4} \\
4(y+5) \cdot \frac{y}{y+5} &= 4(y+5) \cdot \left(\frac{-5}{y+5} + \frac{5}{4}\right) \\
4y &= 4 \cdot (-5) + (y+5) \cdot 5 \\
4y &= -20 + 5y + 25 \\
20 - 25 &= 5y - 4y \\
y &= -5
\end{aligned}$$

Since the denominators of $\dfrac{y}{y+5}$ and $\dfrac{-5}{y+5}$ are equal to zero at $y = -5$, it follows that the equation has no solution.

6. Solve the equation $\dfrac{11}{x^2+5x+4} - \dfrac{3}{x+4} = \dfrac{1}{x+1}$.

Solution: We have

$$\frac{11}{x^2+5x+4} - \frac{3}{x+4} = \frac{1}{x+1}$$
$$\frac{11}{(x+4)(x+1)} - \frac{3}{x+4} = \frac{1}{x+1}$$
$$(x+4)(x+1)\cdot\left(\frac{11}{(x+4)(x+1)} - \frac{3}{x+4}\right) = (x+4)(x+1)\cdot\frac{1}{x+1}$$
$$(x+4)(x+1)\cdot\frac{11}{(x+4)(x+1)} - (x+4)(x+1)\cdot\frac{3}{x+4} = (x+4)(x+1)\cdot\frac{1}{x+1}$$
$$11-(x+1)\cdot 3 = (x+4)\cdot 1$$
$$11-3x-3 = x+4$$
$$8-3x = x+4$$
$$8-3x-8 = x+4-8$$
$$-3x = x-4$$
$$-3x-x = x-4-x$$
$$-4x = -4$$
$$\frac{-4x}{-4} = \frac{-4}{-4}$$
$$x = 1$$

In short,

$$\frac{11}{x^2+5x+4} - \frac{3}{x+4} = \frac{1}{x+1}$$
$$\frac{11}{(x+4)(x+1)} - \frac{3}{x+4} = \frac{1}{x+1}$$
$$(x+4)(x+1)\cdot\left(\frac{11}{(x+4)(x+1)} - \frac{3}{x+4}\right) = (x+4)(x+1)\cdot\frac{1}{x+1}$$
$$11-(x+1)\cdot 3 = (x+4)\cdot 1$$
$$11-3x-3 = x+4$$
$$-3x-x = 4-11+3$$
$$-4x = -4$$
$$x = 1$$

7. Solve for M the equation $F = G\dfrac{mM}{r^2}$.

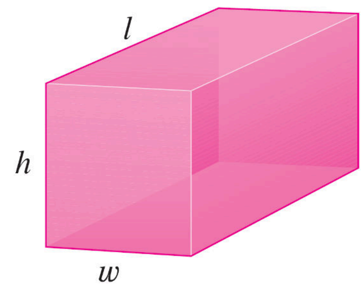
Solution: We have

$$F = \left(\frac{Gm}{r^2}\right) M \quad \Longrightarrow \quad \left(\frac{r^2}{Gm}\right) F = \left(\frac{r^2}{Gm}\right)\left(\frac{Gm}{r^2}\right) M \quad \Longrightarrow \quad \frac{r^2 F}{Gm} = M$$

The solution is $M = \dfrac{r^2 F}{Gm}$.

8. The surface area A of the closed rectangular box can be calculated from the length l, the width w, and the height h according to the formula

$$A = 2lw + 2wh + 2lh$$



Solve for w in terms of the other variables in this equation.

Solution: We have

$$\begin{aligned} A &= 2lw + 2wh + 2lh \\ A &= (2l + 2h)w + 2lh \\ A - 2lh &= (2l + 2h)w \\ \frac{A - 2lh}{2l + 2h} &= w \end{aligned}$$

The solution is $w = \dfrac{A - 2lh}{2l + 2h}$.

Quadratic Equations

Quadratic Equations

A **quadratic equation** is an equation of the form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers with $a \neq 0$.

EXAMPLES:

1. Solve each equation:

(a) $x^2 = 0$ (b) $x^2 = 1$ (c) $x^2 = 4$ (d) $x^2 = 5$ (e) $(x - 4)^2 = 7$

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Solution:

(a) We have $x = 0$.

(b) We have (see the Appendix) $x = \pm 1$.

(c) We have (see the Appendix) $x = \pm 2$ (which is $\pm\sqrt{4}$).

(d) We have (see the Appendix) $x = \pm\sqrt{5}$.

(e) We have

$$(x-4)^2 = 7$$

$$x - 4 = \pm\sqrt{7}$$

$$\left[x - 4 + 4 = \pm\sqrt{7} + 4\right]$$

$$x = 4 \pm \sqrt{7}$$

The solutions are $x = 4 - \sqrt{7}$ and $x = 4 + \sqrt{7}$.

2. Solve the equation $x^2 + 5x = 24$.

Solution: We have

$$x^2 + 5x = 24$$

$$x^2 + 5x - 24 = 0$$

$$(x-3)(x+8) = 0$$

$$x - 3 = 0 \quad \text{or} \quad x + 8 = 0$$

$$x = 3 \quad \text{or} \quad x = -8$$

The solutions are $x = 3$ and $x = -8$.

3. Solve each equation:

(a) $x^2 - 8x + 13 = 0$ (b) $3x^2 - 12x + 7 = 0$

3. Solve each equation:

(a) $x^2 - 8x + 13 = 0$ (b) $3x^2 - 12x + 7 = 0$

Solution:

(a) We have

$$\begin{aligned}
\mathbf{x^2 - 8x + 13} &= \mathbf{0} \\
x^2 - 8x + 13 - 13 &= 0 - 13 \\
x^2 - 8x &= -13 \\
\mathbf{x^2 - 2x \cdot 4} &= \mathbf{-13} \\
\mathbf{x^2 - 2x \cdot 4 + 4^2} &= \mathbf{-13 + 4^2} \\
\mathbf{(x - 4)^2} &= \mathbf{3} \\
\mathbf{x - 4} &= \pm\sqrt{\mathbf{3}} \\
x - 4 + 4 &= \pm\sqrt{3} + 4 \\
\mathbf{x} &= \mathbf{4} \pm \sqrt{\mathbf{3}}
\end{aligned}$$

In short,

$$\begin{aligned}
x^2 - 8x + 13 &= 0 \\
x^2 - 2x \cdot 4 &= -13 \\
x^2 - 2x \cdot 4 + 4^2 &= -13 + 4^2 \\
(x - 4)^2 &= 3 \\
x - 4 &= \pm\sqrt{3} \\
x &= 4 \pm \sqrt{3}
\end{aligned}$$

The solutions are $x = 4 - \sqrt{3}$ and $x = 4 + \sqrt{3}$.

(b) We have

$$\begin{aligned}
\mathbf{3x^2 - 12x + 7} &= \mathbf{0} \\
3x^2 - 12x + 7 - 7 &= 0 - 7 \\
\mathbf{3x^2 - 12x} &= \mathbf{-7} \\
\frac{3x^2 - 12x}{3} &= \frac{-7}{3} \\
\frac{3x^2}{3} - \frac{12x}{3} &= \frac{-7}{3} \\
\mathbf{x^2 - 4x} &= -\frac{\mathbf{7}}{\mathbf{3}} \\
x^2 - 2x \cdot 2 &= -\frac{7}{3} \\
\mathbf{x^2 - 2x \cdot 2 + 2^2} &= -\frac{\mathbf{7}}{\mathbf{3}} + \mathbf{2^2} \\
\mathbf{(x-2)^2} &= \left\{ \frac{-7}{3} + 4 = \frac{-7}{3} + \frac{4}{1} = \frac{-7}{3} + \frac{4 \cdot 3}{1 \cdot 3} = \frac{-7}{3} + \frac{12}{3} = \frac{-7+12}{3} \right\} = \frac{\mathbf{5}}{\mathbf{3}} \\
\mathbf{x - 2} &= \pm\sqrt{\frac{\mathbf{5}}{\mathbf{3}}} \\
x - 2 + 2 &= \pm\sqrt{\frac{5}{3}} + 2 \\
\mathbf{x} &= \mathbf{2} \pm \sqrt{\frac{\mathbf{5}}{\mathbf{3}}}
\end{aligned}$$

In short,

$$\begin{aligned}
3x^2 - 12x + 7 &= 0 \\
3x^2 - 12x &= -7 \\
x^2 - 4x &= -\frac{7}{3} \\
x^2 - 2x \cdot 2 + 2^2 &= -\frac{7}{3} + 2^2 \\
(x-2)^2 &= \frac{5}{3} \\
x - 2 &= \pm\sqrt{\frac{5}{3}} \\
x &= 2 \pm \sqrt{\frac{5}{3}}
\end{aligned}$$

The solutions are $x = 2 - \sqrt{\frac{5}{3}}$ and $x = 2 + \sqrt{\frac{5}{3}}$.

The Quadratic Formula

The roots of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Proof: We have

$$\mathbf{ax^2 + bx + c = 0}$$

$$ax^2 + bx + c - c = 0 - c$$

$$\mathbf{ax^2 + bx = -c}$$

$$\frac{ax^2 + bx}{a} = \frac{-c}{a}$$

$$\frac{ax^2}{a} + \frac{bx}{a} = \frac{-c}{a}$$

$$\mathbf{x^2 + \frac{b}{a}x = -\frac{c}{a}}$$

$$x^2 + 2x \cdot \frac{b}{2a} = -\frac{c}{a}$$

$$\mathbf{x^2 + 2x \cdot \frac{b}{2a} + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2}$$

$$\mathbf{\left(x + \frac{b}{2a}\right)^2} = \left\{\frac{-c}{a} + \frac{b^2}{(2a)^2} = \frac{-c}{a} + \frac{b^2}{2^2a^2} = \frac{-c}{a} + \frac{b^2}{4a^2} = \frac{-4ac}{4a^2} + \frac{b^2}{4a^2}\right\} = \mathbf{\frac{-4ac + b^2}{4a^2}}$$

$$\mathbf{x + \frac{b}{2a}} = \left\{\pm\sqrt{\frac{-4ac + b^2}{4a^2}} = \pm\frac{\sqrt{-4ac + b^2}}{\sqrt{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4}\sqrt{a^2}}\right\} = \mathbf{\pm\frac{\sqrt{b^2 - 4ac}}{2a}}$$

$$x + \frac{b}{2a} - \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} - \frac{b}{2a}$$

$$\mathbf{x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

In short,

$$
\begin{aligned}
ax^2+bx+c&=0\\
ax^2+bx&=-c\\
x^2+\frac{b}{a}x&=-\frac{c}{a}\\
x^2+2x\cdot\frac{b}{2a}+\left(\frac{b}{2a}\right)^2&=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2\\
\left(x+\frac{b}{2a}\right)^2&=\frac{-4ac+b^2}{4a^2}\\
x+\frac{b}{2a}&=\pm\frac{\sqrt{b^2-4ac}}{2a}\\
x&=\frac{-b\pm\sqrt{b^2-4ac}}{2a}
\end{aligned}
$$

EXAMPLES:

1. Solve the equation $3x^2-4x-5=0$.

Solution: We first rewrite the equation as $3x^2+(-4)x+(-5)=0$. Here $a=3,\ b=-4$, and $c=-5$. Therefore by the quadratic formula,

$$
\begin{aligned}
x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-4)\pm\sqrt{(-4)^2-4\cdot 3\cdot(-5)}}{2\cdot 3}=\frac{4\pm\sqrt{16+60}}{2\cdot 3}&=\frac{4\pm\sqrt{76}}{2\cdot 3}\\
&=\frac{4\pm\sqrt{4\cdot 19}}{2\cdot 3}\\
&=\frac{4\pm\sqrt{4}\sqrt{19}}{2\cdot 3}\\
&=\frac{2\cdot 2\pm 2\sqrt{19}}{2\cdot 3}\\
&=\frac{2(2\pm\sqrt{19})}{2\cdot 3}\\
&=\frac{2\pm\sqrt{19}}{3}
\end{aligned}
$$

In short,

$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4\cdot 3\cdot(-5)}}{2\cdot 3}=\frac{4\pm\sqrt{76}}{2\cdot 3}=\frac{4\pm\sqrt{4\cdot 19}}{2\cdot 3}=\frac{2\cdot 2\pm 2\sqrt{19}}{2\cdot 3}=\frac{2\pm\sqrt{19}}{3}$$

2. Solve the equation $x^2 = 4$.

Solution: We first rewrite the equation as

$$1 \cdot x^2 + 0 \cdot x + (-4) = 0$$

Here $a = 1,\ b = 0$, and $c = -4$. Therefore by the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{0 \pm \sqrt{0^2 - 4 \cdot 1 \cdot (-4)}}{2 \cdot 1} = \frac{\pm\sqrt{0 + 16}}{2} = \frac{\pm\sqrt{16}}{2} = \frac{\pm 4}{2} = \pm 2$$

3. Find all solutions of each equation.

(a) $3x^2 - 5x - 1 = 0$

(b) $4x^2 + 12x + 9 = 0$

(c) $x^2 + 2x + 2 = 0$

3. Find all solutions of each equation.

(a) $3x^2 - 5x - 1 = 0$ (b) $4x^2 + 12x + 9 = 0$ (c) $x^2 + 2x + 2 = 0$

Solution:

(a) We first rewrite the equation as $3x^2 + (-5)x + (-1) = 0$. Here $a = 3$, $b = -5$, and $c = -1$. Therefore by the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 3 \cdot (-1)}}{2 \cdot 3} = \frac{5 \pm \sqrt{25 + 12}}{6} = \frac{5 \pm \sqrt{37}}{6}$$

(b) In this quadratic equation $a = 4$, $b = 12$, and $c = 9$. Therefore by the quadratic formula,

$$\begin{aligned} x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-12 \pm \sqrt{12^2 - 4 \cdot 4 \cdot 9}}{2 \cdot 4} = \frac{-12 \pm \sqrt{144 - 144}}{8} &= \frac{-12 \pm \sqrt{0}}{8} \\ &= \frac{-12 \pm 0}{8} \\ &= \frac{-12}{8} \\ &= -\frac{3}{2} \end{aligned}$$

(c) In this quadratic equation $a = 1$, $b = 2$, and $c = 2$. Therefore by the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2}}{2 \cdot 1} = \frac{-2 \pm \sqrt{4 - 8}}{2} = \frac{-2 \pm \sqrt{-4}}{2}$$

Since the square of any real number is nonnegative, $\sqrt{-4}$ is undefined in the real number system. The equation has no real solution.

The Discriminant

The **discriminant** of the general quadratic $ax^2 + bx + c = 0$ $(a \neq 0)$ is $D = b^2 - 4ac$.

1. If $D > 0$, then the equation has two distinct real solutions.

2. If $D = 0$, then the equation has exactly one real solution.

3. If $D < 0$, then the equation has no real solution.

EXAMPLES:

1. Use the discriminant to determine how many real solutions each equation has.

(a) $x^2 + 4x - 1 = 0$ (b) $4x^2 - 12x + 9 = 0$ (c) $\frac{1}{3}x^2 - 2x + 4 = 0$

1. Use the discriminant to determine how many real solutions each equation has.

(a) $x^2 + 4x - 1 = 0$ (b) $4x^2 - 12x + 9 = 0$ (c) $\frac{1}{3}x^2 - 2x + 4 = 0$

Solution:

(a) We first rewrite the equation as $1 \cdot x^2 + 4x + (-1) = 0$. Here $a = 1$, $b = 4$, and $c = -1$. Therefore the discriminant is

$$D = b^2 - 4ac = 4^2 - 4 \cdot 1 \cdot (-1) = 16 + 4 = 20 > 0$$

so the equation has two distinct real solutions.

(b) We first rewrite the equation as $4x^2 + (-12)x + 9 = 0$. Here $a = 4$, $b = -12$, and $c = 9$. Therefore the discriminant is

$$D = b^2 - 4ac = (-12)^2 - 4 \cdot 4 \cdot 9 = 144 - 144 = 0$$

so the equation has exactly one real solution.

(c) We first rewrite the equation as $\frac{1}{3}x^2 + (-2)x + 4 = 0$. Here $a = \frac{1}{3}$, $b = -2$, and $c = 4$. Therefore the discriminant is

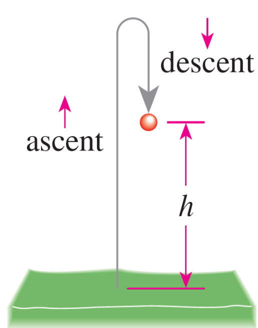
$$D = b^2 - 4ac$$
$$= (-2)^2 - 4 \cdot \frac{1}{3} \cdot 4 = \left\{ 4 - \frac{4 \cdot 1 \cdot 4}{3} = \frac{4}{1} - \frac{16}{3} = \frac{4 \cdot 3}{1 \cdot 3} - \frac{16}{3} = \frac{12}{3} - \frac{16}{3} = \frac{12 - 16}{3} \right\} = \frac{-4}{3} < 0$$

so the equation has no real solution.

2. An object thrown or fired straight upward at an initial speed of v_0 ft/s will reach a height of h feet after t seconds, where h and t are related by the formula

$$h = -16t^2 + v_0 t$$

Suppose that a bullet is shot straight upward with an initial speed of 800 ft/s. Its path is shown in the Figure below.



(a) When does the bullet fall back to ground level?

(b) When does it reach a height of 6400 ft?

(c) When does it reach a height of 2 mi?

(d) How high is the highest point the bullet reaches?

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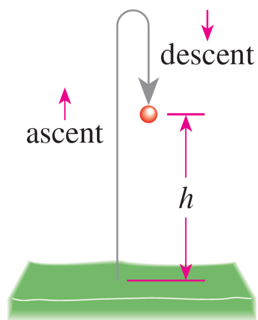


(a) When does the bullet fall back to ground level?

Solution: Since the initial speed in this case is $v_0 = 800$ ft/s, the formula is

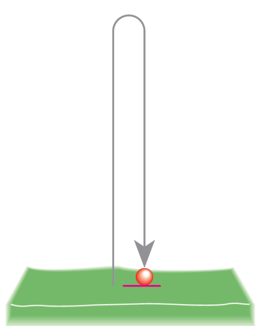
$$h = -16t^2 + 800t$$

Ground level corresponds to $h = 0$, so we must solve the equation

$$0 = -16t^2 + 800t$$

$$0 = -16t(t - 50)$$

Thus, $t = 0$ or $t = 50$. This means the bullet starts ($t = 0$) at ground level and returns to ground level after 50 s.

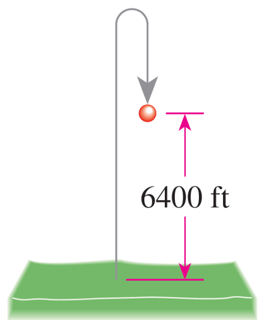


(b) When does it reach a height of 6400 ft?

Solution: Setting $h = 6400$ in $h = -16t^2 + 800t$ gives

$$6400 = -16t^2 + 800t$$
$$16t^2 - 800t + 6400 = 0$$
$$t^2 - 50t + 400 = 0$$
$$(t-10)(t-40) = 0$$
$$t = 10 \quad \text{or} \quad t = 40$$

The bullet reaches 6400 ft after 10 s (on its ascent) and again after 40 s (on its descent to earth).



(c) When does it reach a height of 2 mi?

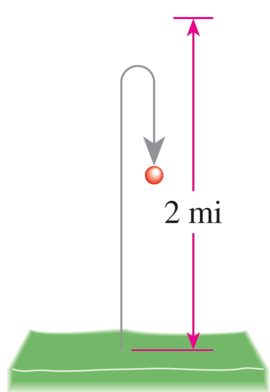
Solution: Two miles is $2 \times 5280 = 10,560$ ft. Setting $h = 10,560$ in $h = -16t^2 + 800t$ gives

$$10,560 = -16t^2 + 800t$$
$$16t^2 - 800t + 10,560 = 0$$
$$t^2 - 50t + 660 = 0$$

The discriminant of this equation is

$$D = b^2 - 4ac = (-50)^2 - 4 \cdot 660 = -140$$

which is negative. Thus, the equation has no real solution. The bullet never reaches a height of 2 mi.



(d) How high is the highest point the bullet reaches?

Solution: Each height the bullet reaches is attained twice, once on its ascent and once on its descent. The only exception is the highest point of its path, which is reached only once. This means that for the highest value of h, the following equation has only one solution for t:

$$h = -16t^2 + 800t$$

$$16t^2 - 800t + h = 0$$

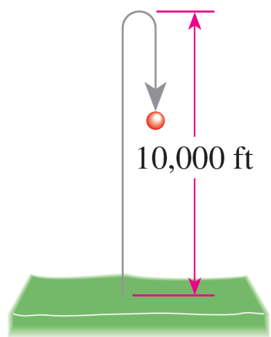
This in turn means that the discriminant D of the equation is 0, and so

$$D = b^2 - 4ac = (-800)^2 - 4 \cdot 16 \cdot h = 0$$

$$640,000 - 64h = 0$$

$$h = 10,000$$

The maximum height reached is 10,000 ft.



Other Types of Equations

EXAMPLES:

1. Solve the equation $x^3 = x$.

Solution 1: We have

$$x^3 = x$$

$$x^3 - x = 0$$

$$x(x^2 - 1) = 0$$

$$x(x-1)(x+1) = 0$$

$$x = 0 \quad \text{or} \quad x - 1 = 0 \quad \text{or} \quad x + 1 = 0$$

$$x = 0 \quad \text{or} \quad x = 1 \quad \text{or} \quad x = -1$$

Solution 2: If $x \neq 0$, then

$$x^3 = x$$

$$\frac{x^3}{x} = \frac{x}{x}$$

$$x^2 = 1$$

$$x = \pm 1$$

Note that $x = 0$ is also a solution of the equation. This gives the same result.

2. Solve the following equations:

(a) $x^6 = 16x^2$ (b) $x^7 = 27x^4$

2. Solve the following equations:

(a) $x^6 = 16x^2$ (b) $x^7 = 27x^4$

Solution 1(a): We have

$$\begin{aligned} x^6 &= 16x^2 \\ x^6 - 16x^2 &= 0 \\ x^2(x^4 - 16) &= 0 \\ x^2\Big((x^2)^2 - 4^2\Big) &= 0 \\ x^2(x^2 - 4)(x^2 + 4) &= 0 \\ x^2(x - 2)(x + 2)(x^2 + 4) &= 0 \end{aligned}$$

Since $x^2 + 4 \neq 0$, we have

$$\begin{aligned} x^2 &= 0 \quad \text{or} \quad x - 2 = 0 \quad \text{or} \quad x + 2 = 0 \\ x &= 0 \quad \text{or} \quad x = 2 \quad \text{or} \quad x = -2 \end{aligned}$$

Solution 2(a): If $x \neq 0$, then

$$\begin{aligned} x^6 &= 16x^2 \\ \frac{x^6}{x^2} &= \frac{16x^2}{x^2} \\ x^4 &= 16 \\ \sqrt[4]{x^4} &= \sqrt[4]{16} \\ |x| &= 2 \\ x &= \pm 2 \end{aligned}$$

Note that $x = 0$ is also a solution of the equation. This gives the same result.

Solution 1(b): We have

$$\begin{aligned} x^7 &= 27x^4 \\ x^7 - 27x^4 &= 0 \\ x^4(x^3 - 27) &= 0 \\ x^4(x^3 - 3^3) &= 0 \\ x^4(x - 3)(x^2 + 3x + 9) &= 0 \end{aligned}$$

Since the discriminant of $x^2 + 3x + 9$ is $D = 3^2 - 4 \cdot 1 \cdot 9 = -27 < 0$, it follows that $x^2 + 3x + 9 \neq 0$. Therefore

$$\begin{aligned} x^4 &= 0 \quad \text{or} \quad x - 3 = 0 \\ x &= 0 \quad \text{or} \quad x = 3 \end{aligned}$$

Solution 2(b): If $x \neq 0$, then

$$\begin{aligned} x^7 &= 27x^4 \\ \frac{x^7}{x^4} &= \frac{27x^4}{x^4} \\ x^3 &= 27 \\ \sqrt[3]{x^3} &= \sqrt[3]{27} \\ x &= 3 \end{aligned}$$

Note that $x = 0$ is also a solution of the equation. This gives the same result.

3. Solve the equation $20a^3 - 12a^2 - 45a + 27 = 0$.

Solution: We have

$$\begin{aligned} 20a^3 - 12a^2 - 45a + 27 &= 0 \\ 4a^2(5a - 3) - 9(5a - 3) &= 0 \\ (5a - 3)(4a^2 - 9) &= 0 \\ (5a - 3)\Big((2a)^2 - 3^2\Big) &= 0 \\ (5a - 3)(2a - 3)(2a + 3) &= 0 \end{aligned}$$

$$\begin{aligned} 5a - 3 &= 0 \quad \text{or} \quad 2a - 3 = 0 \quad \text{or} \quad 2a + 3 = 0 \\ 5a &= 3 \quad \text{or} \quad 2a = 3 \quad \text{or} \quad 2a = -3 \\ a &= \frac{3}{5} \quad \text{or} \quad a = \frac{3}{2} \quad \text{or} \quad a = -\frac{3}{2} \end{aligned}$$

4. Solve the equation $\dfrac{3}{x} + \dfrac{5}{x+2} = 2$.

Solution: We have

$$\begin{aligned}
\frac{3}{x} + \frac{5}{x+2} &= 2 \\
\left(\frac{3}{x} + \frac{5}{x+2}\right) \cdot x(x+2) &= 2 \cdot x(x+2) \\
\frac{3}{x} \cdot x(x+2) + \frac{5}{x+2} \cdot x(x+2) &= 2x^2 + 4x \\
3(x+2) + 5x &= 2x^2 + 4x \\
3x + 6 + 5x &= 2x^2 + 4x \\
0 &= 2x^2 + 4x - 3x - 6 - 5x \\
0 &= 2x^2 - 4x - 6 \\
0 &= x^2 - 2x - 3
\end{aligned}$$

To solve $x^2 - 2x - 3 = 0$ we can either factor

$$\begin{aligned}
x^2 - 2x - 3 &= 0 \\
(x-3)(x+1) &= 0 \\
x - 3 = 0 \quad &\text{or} \quad x + 1 = 0 \\
x = 3 \quad &\text{or} \quad x = -1
\end{aligned}$$

or use the quadratic formula with $a = 1, b = -2$, and $c = -3$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-3)}}{2 \cdot 1} = \frac{2 \pm \sqrt{4 + 12}}{2} = \frac{2 \pm \sqrt{16}}{2} = \frac{2 \pm 4}{2}$$

so

$$x = \frac{2+4}{2} = \frac{6}{2} = 3 \qquad \text{or} \qquad x = \frac{2-4}{2} = \frac{-2}{2} = -1$$

The values $x = 3$ and $x = -1$ are only potential solutions. We must check them to see if they satisfy the original equation.

Check: $x = 3$

$$\begin{aligned}
\frac{3}{x} + \frac{5}{x+2} &= 2 \\
\frac{3}{3} + \frac{5}{3+2} &\stackrel{?}{=} 2 \\
1 + 1 = 2 &\quad \checkmark\text{TRUE}
\end{aligned}$$

Check: $x = -1$

$$\begin{aligned}
\frac{3}{x} + \frac{5}{x+2} &= 2 \\
\frac{3}{-1} + \frac{5}{-1+2} &\stackrel{?}{=} 2 \\
-3 + 5 = 2 &\quad \checkmark\text{TRUE}
\end{aligned}$$

We see that both $x = 3$ and $x = -1$ are the solutions of the equation $\dfrac{3}{x} + \dfrac{5}{x+2} = 2$.

5. Solve the equation $2x = 1 - \sqrt{2-x}$.

Solution: We have

$$\begin{aligned} 2x &= 1 - \sqrt{2-x} \\ 2x - 1 &= -\sqrt{2-x} \\ (2x-1)^2 &= (-\sqrt{2-x})^2 \\ (2x)^2 - 2\cdot 2x \cdot 1 + 1^2 &= 2 - x \\ 4x^2 - 4x + 1 &= 2 - x \\ 4x^2 - 4x + 1 - 2 + x &= 0 \\ 4x^2 - 3x - 1 &= 0 \end{aligned}$$

To solve $4x^2 - 3x - 1 = 0$ we can either factor

$$\begin{aligned} 4x^2 - 3x - 1 &= 0 \\ (4x+1)(x-1) &= 0 \\ 4x + 1 = 0 \quad &\text{or} \quad x - 1 = 0 \\ x = -\frac{1}{4} \quad &\text{or} \quad x = 1 \end{aligned}$$

or use the quadratic formula with $a = 4, b = -3$, and $c = -1$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4\cdot 4 \cdot (-1)}}{2\cdot 4} = \frac{3 \pm \sqrt{9+16}}{8} = \frac{3 \pm \sqrt{25}}{8} = \frac{3 \pm 5}{8}$$

so

$$x = \frac{3-5}{8} = \frac{-2}{8} = -\frac{1}{4} \qquad \text{or} \qquad x = \frac{3+5}{8} = \frac{8}{8} = 1$$

The values $x = -\frac{1}{4}$ and $x = 1$ are only potential solutions. We must check them to see if they satisfy the original equation.

Check: $x = -\frac{1}{4}$

$$\begin{aligned} 2x &= 1 - \sqrt{2-x} \\ 2\cdot\left(-\frac{1}{4}\right) &\stackrel{?}{=} 1 - \sqrt{2 - \left(-\frac{1}{4}\right)} \\ -\frac{1}{2} &\stackrel{?}{=} 1 - \sqrt{2 + \frac{1}{4}} \\ -\frac{1}{2} &\stackrel{?}{=} 1 - \sqrt{\frac{9}{4}} \\ -\frac{1}{2} &= 1 - \frac{3}{2} \quad \checkmark\text{TRUE} \end{aligned}$$

Check: $x = 1$

$$\begin{aligned} 2x &= 1 - \sqrt{2-x} \\ 2\cdot 1 &\stackrel{?}{=} 1 - \sqrt{2-1} \\ 2 &\stackrel{?}{=} 1 - \sqrt{1} \\ 2 &= 1 - 1 \quad \text{FALSE} \end{aligned}$$

We see that $x = -\frac{1}{4}$ is a solution but $x = 1$ is not. So, the only solution is $x = -\frac{1}{4}$.

6. Solve the equation $\sqrt{6+2x} - \sqrt{x+7} = -2$.

6. Solve the equation $\sqrt{6+2x} - \sqrt{x+7} = -2$.

Solution 1: We have

$$\begin{aligned}
\sqrt{6+2x} - \sqrt{x+7} &= -2 \\
\sqrt{6+2x} &= -2 + \sqrt{x+7} \\
(\sqrt{6+2x})^2 &= (-2+\sqrt{x+7})^2 \\
6+2x &= (-2)^2 + 2(-2)\sqrt{x+7} + (\sqrt{x+7})^2 \\
6+2x &= 4 - 4\sqrt{x+7} + x + 7 \\
6+2x-4-x-7 &= -4\sqrt{x+7} \\
x-5 &= -4\sqrt{x+7} \\
(x-5)^2 &= (-4\sqrt{x+7})^2 \\
x^2 - 2\cdot x \cdot 5 + 5^2 &= 16(x+7) \\
x^2 - 10x + 25 &= 16x + 112 \\
x^2 - 10x + 25 - 16x - 112 &= 0 \\
x^2 - 26x - 87 &= 0
\end{aligned}$$

To solve $x^2 - 26x - 87 = 0$ we can either factor

$$\begin{aligned}
x^2 - 26x - 87 &= 0 \\
(x-29)(x+3) &= 0 \\
x - 29 = 0 \quad &\text{or} \quad x+3=0 \\
x = 29 \quad &\text{or} \quad x = -3
\end{aligned}$$

or use the quadratic formula with $a = 1, b = -26$, and $c = -87$:

$$\begin{aligned}
x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-26) \pm \sqrt{(-26)^2 - 4\cdot 1\cdot(-87)}}{2\cdot 1} = \frac{26 \pm \sqrt{676+348}}{2} &= \frac{26 \pm \sqrt{1024}}{2} \\
&= \frac{26 \pm 32}{2}
\end{aligned}$$

so

$$x = \frac{26+32}{2} = \frac{58}{2} = 29 \qquad \text{or} \qquad x = \frac{26-32}{2} = \frac{-6}{2} = -3$$

The values $x = 29$ and $x = -3$ are only potential solutions. We must check them to see if they satisfy the original equation.

Check: $x = 29$

$$\begin{aligned}
\sqrt{6+2x} - \sqrt{x+7} &= -2 \\
\sqrt{6+2\cdot 29} - \sqrt{29+7} &\stackrel{?}{=} -2 \\
\sqrt{64} - \sqrt{36} &\stackrel{?}{=} -2 \\
8 - 6 &= -2 \ \text{ FALSE}
\end{aligned}$$

Check: $x = -3$

$$\begin{aligned}
\sqrt{6+2x} - \sqrt{x+7} &= -2 \\
\sqrt{6+2\cdot(-3)} - \sqrt{-3+7} &\stackrel{?}{=} -2 \\
\sqrt{0} - \sqrt{4} &\stackrel{?}{=} -2 \\
0 - 2 &= -2 \ \checkmark\text{TRUE}
\end{aligned}$$

We see that $x = -3$ is a solution but $x = 29$ is not. So, the only solution is $x = -3$.

Solution 2: We have

$$\begin{aligned}
\sqrt{6+2x}-\sqrt{x+7} &= -2\\
(\sqrt{6+2x}-\sqrt{x+7})^2 &= (-2)^2\\
(\sqrt{6+2x})^2-2\sqrt{6+2x}\sqrt{x+7}+(\sqrt{x+7})^2 &= 4\\
6+2x-2\sqrt{(6+2x)(x+7)}+x+7 &= 4\\
-2\sqrt{\underbrace{(6+2x)(x+7)}_{6x+42+2x^2+14x=2x^2+20x+42}} &= \underbrace{4-6-2x-x-7}_{-3x-9}\\
-2\sqrt{2x^2+20x+42} &= -3x-9\\
2\sqrt{2x^2+20x+42} &= 3x+9\\
(2\sqrt{2x^2+20x+42})^2 &= (3x+9)^2\\
4(2x^2+20x+42) &= (3x)^2+2\cdot 3x\cdot 9+9^2\\
8x^2+80x+168 &= 9x^2+54x+81\\
0 &= 9x^2+54x+81-8x^2-80x-168\\
0 &= x^2-26x-87
\end{aligned}$$

and the same result follows.

7. Solve the equation $x^4-8x^2+8=0$.

Solution: Setting $W=x^2$, we get

$$\begin{aligned}
x^4-8x^2+8 &= 0\\
(x^2)^2-8x^2+8 &= 0\\
W^2-8W+8 &= 0
\end{aligned}$$

To solve this equation, we use the quadratic formula with $a=1, b=-8$, and $c=8$:

$$\begin{aligned}
W &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-(-8)\pm\sqrt{(-8)^2-4\cdot 8}}{2} = \frac{8\pm\sqrt{64-32}}{2}\\
&= \frac{8\pm\sqrt{32}}{2} = \frac{8\pm\sqrt{16\cdot 2}}{2} = \frac{8\pm\sqrt{16}\sqrt{2}}{2} = \frac{8\pm 4\sqrt{2}}{2} = \frac{8}{2}\pm\frac{4\sqrt{2}}{2} = 4\pm 2\sqrt{2}
\end{aligned}$$

so

$$x^2 = 4\pm 2\sqrt{2} \quad\Longrightarrow\quad x=\pm\sqrt{4\pm 2\sqrt{2}}$$

It follows that there are four solutions:

$$\sqrt{4+2\sqrt{2}} \qquad \sqrt{4-2\sqrt{2}} \qquad -\sqrt{4+2\sqrt{2}} \qquad -\sqrt{4-2\sqrt{2}}$$

8. Solve the equation $x^{1/3}+x^{1/6}-2=0$.

8. Solve the equation $x^{1/3} + x^{1/6} - 2 = 0$.

Solution: Setting $W = x^{1/6}$, we get

$$\begin{aligned} x^{1/3} + x^{1/6} - 2 &= 0 \\ (x^{1/6})^2 + x^{1/6} - 2 &= 0 \\ W^2 + W - 2 &= 0 \\ (W-1)(W+2) &= 0 \end{aligned}$$

$$\begin{array}{rcr} W - 1 = 0 & \text{or} & W + 2 = 0 \\ W = 1 & & W = -2 \\ x^{1/6} = 1 & & x^{1/6} = -2 \\ (x^{1/6})^6 = 1^6 & & (x^{1/6})^6 = (-2)^6 \\ x = 1 & & x = 64 \end{array}$$

The values $x = 1$ and $x = 64$ are only potential solutions. We must check them to see if they satisfy the original equation.

Check: $x = 1$

$$\begin{aligned} x^{1/3} + x^{1/6} - 2 &= 0 \\ 1^{1/3} + 1^{1/6} - 2 &\stackrel{?}{=} 0 \\ 1 + 1 - 2 &= 0 \quad \checkmark\text{TRUE} \end{aligned}$$

Check: $x = 64$

$$\begin{aligned} x^{1/3} + x^{1/6} - 2 &= 0 \\ 64^{1/3} + 64^{1/6} - 2 &\stackrel{?}{=} 0 \\ 4 + 2 - 2 &= 0 \quad \text{FALSE} \end{aligned}$$

We see $x = 1$ is a solution but $x = 64$ is not. So, the only solution is $x = 1$.

Absolute Value Equations

EXAMPLES:

1. Solve the equation $|2x - 5| = 3$.

Solution: By the definition of absolute value, $|2x - 5| = 3$ is equivalent to

$$\begin{array}{rcr} 2x - 5 = 3 & \quad\text{or}\quad & 2x - 5 = -3 \\ 2x = 8 & & 2x = 2 \\ x = 4 & & x = 1 \end{array}$$

The solutions are $x = 4$ and $x = 1$.

2. Solve the equation $5|2 - 4x| + 3 = 53$.

Solution: We have

$$5|2 - 4x| + 3 = 53$$

$$5|2 - 4x| = \underbrace{53 - 3}_{50}$$

$$|2 - 4x| = \frac{50}{5} = 10$$

By the definition of absolute value, $|2 - 4x| = 10$ is equivalent to

$$\begin{array}{rcr} 2 - 4x = 10 & \quad\text{or}\quad & 2 - 4x = -10 \\ -4x = \underbrace{10 - 2}_{8} & & -4x = \underbrace{-10 - 2}_{-12} \\ x = \dfrac{8}{-4} = -2 & & x = \dfrac{-12}{-4} = 3 \end{array}$$

The solutions are $x = -2$ and $x = 3$.

3. Solve the equation $|-x + 7| = |2x - 4|$.

Solution: By the definition of absolute value, $|-x + 7| = |2x - 4|$ is equivalent to

$$\begin{array}{rcr} -x + 7 = 2x - 4 & \quad\text{or}\quad & -x + 7 = -(2x - 4) \\ -x - 2x = -4 - 7 & & -x + 7 = -2x + 4 \\ -3x = -11 & & -x + 2x = 4 - 7 \\ x = \dfrac{-11}{-3} = \dfrac{11}{3} & & x = -3 \end{array}$$

The solutions are $x = \frac{11}{3}$ and $x = -3$.

Appendix

1. Solve the equation $x^2 = 1$.

Solution 1: We have

$$\begin{aligned} x^2 &= 1 \\ x^2 - 1 &= 0 \\ x^2 - 1^2 &= 0 \\ (x+1)(x-1) &= 0 \\ x+1 = 0 \quad &\text{or} \quad x-1=0 \\ x=-1 \quad &\text{or} \quad x=1 \end{aligned}$$

The solutions are $x = -1$ and $x = 1$.

Solution 2: We have

$$\begin{aligned} x^2 &= 1 \\ \sqrt{x^2} &= \sqrt{1} \\ |x| &= 1 \\ x &= \pm 1 \end{aligned}$$

2. Solve the equation $x^2 = 4$.

Solution 1: We have

$$\begin{aligned} x^2 &= 4 \\ x^2 - 4 &= 0 \\ x^2 - 2^2 &= 0 \\ (x+2)(x-2) &= 0 \\ x+2 = 0 \quad &\text{or} \quad x-2=0 \\ x=-2 \quad &\text{or} \quad x=2 \end{aligned}$$

The solutions are $x = -2$ and $x = 2$.

Solution 2: We have

$$\begin{aligned} x^2 &= 4 \\ \sqrt{x^2} &= \sqrt{4} \\ |x| &= 2 \\ x &= \pm 2 \end{aligned}$$

3. Solve the equation $x^2 = 5$.

Solution 1: We have

$$\begin{aligned} x^2 &= 5 \\ x^2 - 5 &= 0 \\ x^2 - (\sqrt{5})^2 &= 0 \\ (x+\sqrt{5})(x-\sqrt{5}) &= 0 \\ x+\sqrt{5} = 0 \quad &\text{or} \quad x-\sqrt{5}=0 \\ x=-\sqrt{5} \quad &\text{or} \quad x=\sqrt{5} \end{aligned}$$

The solutions are $x = -\sqrt{5}$ and $x = \sqrt{5}$.

Solution 2: We have

$$\begin{aligned} x^2 &= 5 \\ \sqrt{x^2} &= \sqrt{5} \\ |x| &= \sqrt{5} \\ x &= \pm\sqrt{5} \end{aligned}$$