Section 1.4 Rational Expressions

The Domain of an Algebraic Expression

The **domain** of an algebraic expression is the set of real numbers that the variable is permitted to have.

EXAMPLES:

1. The domain of $\dfrac{2x+4}{x-3}$ is all real numbers except 3. We can write this in set notation as

$$\{x \mid x \neq 3\} \quad \text{or} \quad (-\infty, 3) \cup (3, \infty)$$

2. The domain of $\dfrac{1-5x}{x^2-3}$ is all real numbers except $\pm\sqrt{3}$:

$$\{x \mid x \neq \pm\sqrt{3}\} \quad \text{or} \quad (-\infty, -\sqrt{3}) \cup (-\sqrt{3}, \sqrt{3}) \cup (\sqrt{3}, \infty)$$

3. The domain of $\dfrac{2x+4}{x^2+3x+2}$ is all real numbers except -1 and -2:

$$\{x \mid x \neq -1 \text{ and } x \neq -2\} \quad \text{or} \quad (-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$$

4. The domain of $\dfrac{2x+4}{x^2+2}$ is all real numbers $\mathbb{R}$: $(-\infty, \infty)$

Simplifying Rational Expressions

To simplify rational expressions, we factor both numerator and denominator and use the following property of fractions:

$$\boxed{\frac{AC}{BC} = \frac{A}{B}}$$

EXAMPLES:

1. $$\frac{x^2-1}{x^2+x-2} = \frac{(x-1)(x+1)}{(x-1)(x+2)} = \frac{x+1}{x+2}$$

2. $$\frac{x^2+2x-3}{x^2+8x+16} \cdot \frac{3x+12}{x-1} = \frac{(x-1)(x+3)}{(x+4)^2} \cdot \frac{3(x+4)}{x-1} = \frac{3(x-1)(x+3)(x+4)}{(x-1)(x+4)^2} = \frac{3(x+3)}{x+4}$$

3. $$\frac{x-4}{x^2-4} \div \frac{x^2-3x-4}{x^2+5x+6} = \frac{x-4}{x^2-4} \cdot \frac{x^2+5x+6}{x^2-3x-4} = \frac{(x-4)(x+2)(x+3)}{(x-2)(x+2)(x-4)(x+1)} = \frac{x+3}{(x-2)(x+1)}$$

4. $$\frac{3}{x-1} + \frac{x}{x+2} = \frac{3(x+2)}{(x-1)(x+2)} + \frac{x(x-1)}{(x-1)(x+2)} = \frac{3x+6+x^2-x}{(x-1)(x+2)} = \frac{x^2+2x+6}{(x-1)(x+2)}$$

5. $$\frac{1}{x^2-1} - \frac{2}{(x+1)^2} = \frac{1}{(x-1)(x+1)} - \frac{2}{(x+1)^2} = \frac{x+1}{(x-1)(x+1)^2} - \frac{2(x-1)}{(x-1)(x+1)^2}$$
$$= \frac{(x+1)-2(x-1)}{(x-1)(x+1)^2} = \frac{x+1-2x+2}{(x-1)(x+1)^2} = \frac{3-x}{(x-1)(x+1)^2}$$

Compound Fractions

A **compound fraction** is a fraction in which the numerator, the denominator, or both are themselves fractional expressions.

EXAMPLES:

1_1. $$\frac{\dfrac{x}{y}+1}{1-\dfrac{y}{x}}=\frac{\dfrac{x}{y}+\dfrac{y}{y}}{\dfrac{x}{x}-\dfrac{y}{x}}=\frac{\dfrac{x+y}{y}}{\dfrac{x-y}{x}}=\frac{x+y}{y}\cdot\frac{x}{x-y}=\frac{x(x+y)}{y(x-y)}$$

1_2. $$\frac{\dfrac{x}{y}+1}{1-\dfrac{y}{x}}=\frac{\left(\dfrac{x}{y}+1\right)xy}{\left(1-\dfrac{y}{x}\right)xy}=\frac{\dfrac{x}{y}\cdot xy+1\cdot xy}{1\cdot xy-\dfrac{y}{x}\cdot xy}=\frac{x^2+xy}{xy-y^2}=\frac{x(x+y)}{y(x-y)}$$

2. $$\frac{\dfrac{1}{a+h}-\dfrac{1}{a}}{h}=\frac{\dfrac{a-(a+h)}{a(a+h)}}{h}=\frac{a-(a+h)}{a(a+h)}\cdot\frac{1}{h}=\frac{a-a-h}{a(a+h)}\cdot\frac{1}{h}=\frac{-h}{a(a+h)}\cdot\frac{1}{h}=\frac{-1}{a(a+h)}$$

3_1. $$\frac{(1+x^2)^{1/2}-x^2(1+x^2)^{-1/2}}{1+x^2}=\frac{(1+x^2)^{-1/2}[(1+x^2)-x^2]}{1+x^2}=\frac{(1+x^2)^{-1/2}}{1+x^2}=\frac{1}{(1+x^2)^{3/2}}$$

3_2. $$\frac{(1+x^2)^{1/2}-x^2(1+x^2)^{-1/2}}{1+x^2}=\frac{(1+x^2)^{1/2}-x^2(1+x^2)^{-1/2}}{1+x^2}\cdot\frac{(1+x^2)^{1/2}}{(1+x^2)^{1/2}}=\frac{(1+x^2)-x^2}{(1+x^2)^{3/2}}$$
$$=\frac{1}{(1+x^2)^{3/2}}$$

Rationalizing the Denominator or the Numerator

EXAMPLES:

1. $$\frac{1}{1+\sqrt{2}}=\frac{1\cdot(1-\sqrt{2})}{(1+\sqrt{2})\cdot(1-\sqrt{2})}=\frac{1-\sqrt{2}}{1^2-(\sqrt{2})^2}=\frac{1-\sqrt{2}}{1-2}=\frac{1-\sqrt{2}}{-1}=\sqrt{2}-1$$

2. $$\frac{\sqrt{4+h}-2}{h}=\frac{(\sqrt{4+h}-2)(\sqrt{4+h}+2)}{h(\sqrt{4+h}+2)}=\frac{(\sqrt{4+h})^2-2^2}{h(\sqrt{4+h}+2)}=\frac{4+h-4}{h(\sqrt{4+h}+2)}=\frac{h}{h(\sqrt{4+h}+2)}$$
$$=\frac{1}{\sqrt{4+h}+2}$$

3. Evaluate
$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}$$

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Solution: We have

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}=\frac{1}{\sqrt{2}+\sqrt{1}}+\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{\sqrt{4}+\sqrt{3}}$$

$$=\frac{\sqrt{2}-\sqrt{1}}{(\sqrt{2}+\sqrt{1})(\sqrt{2}-\sqrt{1})}+\frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}+\frac{\sqrt{4}-\sqrt{3}}{(\sqrt{4}+\sqrt{3})(\sqrt{4}-\sqrt{3})}$$

$$=\frac{\sqrt{2}-\sqrt{1}}{(\sqrt{2})^2-(\sqrt{1})^2}+\frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3})^2-(\sqrt{2})^2}+\frac{\sqrt{4}-\sqrt{3}}{(\sqrt{4})^2-(\sqrt{3})^2}$$

$$=\frac{\sqrt{2}-\sqrt{1}}{2-1}+\frac{\sqrt{3}-\sqrt{2}}{3-2}+\frac{\sqrt{4}-\sqrt{3}}{4-3}=\frac{\sqrt{2}-\sqrt{1}}{1}+\frac{\sqrt{3}-\sqrt{2}}{1}+\frac{\sqrt{4}-\sqrt{3}}{1}$$

$$=\sqrt{2}-\sqrt{1}+\sqrt{3}-\sqrt{2}+\sqrt{4}-\sqrt{3}=-\sqrt{1}+\sqrt{4}=1$$

REMARK: In the same way one can prove that

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\ldots+\frac{1}{\sqrt{n-1}+\sqrt{n}}=\sqrt{n}-1$$

Rationalizing the Denominator or the Numerator

Don't make the mistake of applying properties of multiplication to the operation of addition. Many of the common errors in algebra involve doing just that. The following table states several properties of multiplication and illustrates the error in applying them to addition.

Correct multiplication property	**Common error with addition**
$(a \cdot b)^2 = a^2 \cdot b^2$	$(a+b)^2 \neq a^2+b^2$
$\sqrt{a \cdot b} = \sqrt{a}\sqrt{b} \quad (a, b \geq 0)$	$\sqrt{a+b} \neq \sqrt{a}+\sqrt{b}$
$\sqrt{a^2 \cdot b^2} = a \cdot b \quad (a, b \geq 0)$	$\sqrt{a^2+b^2} \neq a+b$
$\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{a \cdot b}$	$\frac{1}{a}+\frac{1}{b} \neq \frac{1}{a+b}$
$\frac{ab}{a} = b$	$\frac{a+b}{a} \neq b$
$a^{-1} \cdot b^{-1} = (a \cdot b)^{-1}$	$a^{-1}+b^{-1} \neq (a+b)^{-1}$