Section 1.3 Algebraic Expressions

Adding and Subtracting Polynomials

POLYNOMIALS

A **polynominal** in the variable x is an expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are real numbers and n is a nonnegative integer. If $a_n \neq 0$, then the polynomial has **degree n**. The monomials $a_k x^k$ that make up the polynomial are called the **terms** of the polynomial.

Polynomial	Type	Terms	Degree
$2x^2 - 3x + 4$	trinomial	$2x^2, -3x, 4$	2
$x^8 + 5x$	binomial	$x^8, 5x$	8
$8 - x + x^2 - \frac{1}{2}x^3$	four terms	$-\frac{1}{2}x^3, x^2, -x, 8$	3
$5x + 1$	binomial	$5x, 1$	1
$9x^5$	monomial	$9x^5$	5
6	monomial	6	0

EXAMPLES:

1. $(x^3 - 6x^2 + 2x + 4) + (x^3 + 5x^2 - 7x) = (x^3 + x^3) + (-6x^2 + 5x^2) + (2x - 7x) + 4$
$= 2x^3 - x^2 - 5x + 4$

2. $(x^3 - 6x^2 + 2x + 4) - (x^3 + 5x^2 - 7x) = x^3 - 6x^2 + 2x + 4 - x^3 - 5x^2 + 7x$
$= (x^3 - x^3) + (-6x^2 - 5x^2) + (2x + 7x) + 4$
$= -11x^2 + 9x + 4$

3. $x^3 - 6x^2 + 2x + 4 - x^3 + 5x^2 - 7x = (x^3 - x^3) + (-6x^2 + 5x^2) + (2x - 7x) + 4$
$= -x^2 - 5x + 4$

4. $5(3t - 4) - (t^2 + 2) - 2t(t - 3) = 15t - 20 - t^2 - 2 - 2t^2 + 6t$
$= (-t^2 - 2t^2) + (15t + 6t) - 20 - 2$
$= -3t^2 + 21t - 22$

Multiplying Algebraic Expressions

EXAMPLES:

1. $$\begin{aligned}(2x+1)(3x-5) &= 2x(3x-5)+1(3x-5)\\ &= (2x\cdot 3x-2x\cdot 5)+(1\cdot 3x-1\cdot 5)\\ &= (6x^2-10x)+(3x-5)\\ &= 6x^2-10x+3x-5\\ &= 6x^2-7x-5\end{aligned}$$

2. $$\begin{aligned}(2x-3)(x^2-5x+4) &= 2x(x^2-5x+4)-3(x^2-5x+4)\\ &= (2x\cdot x^2-2x\cdot 5x+2x\cdot 4)-(3\cdot x^2-3\cdot 5x+3\cdot 4)\\ &= (2x^3-10x^2+8x)-(3x^2-15x+12)\\ &= 2x^3-10x^2+8x-3x^2+15x-12\\ &= 2x^3-13x^2+23x-12\end{aligned}$$

Special Product Formulas

SPECIAL PRODUCT FORMULAS

If A and B are any real numbers or algebraic expressions, then

1. $(A+B)(A-B)=A^2-B^2$	Sum and difference of same terms
2. $(A+B)^2=A^2+2AB+B^2$	Square of a sum
3. $(A-B)^2=A^2-2AB+B^2$	Square of a difference
4. $(A+B)^3=A^3+3A^2B+3AB^2+B^3$	Cube of a sum
5. $(A-B)^3=A^3-3A^2B+3AB^2-B^3$	Cube of a difference

EXAMPLES:

1. $(3x+5)^2=(3x)^2+2(3x)(5)+5^2=9x^2+30x+25$

2. $(x^2-2)^3=(x^2)^3-3(x^2)^2(2)+3(x^2)(2)^2-2^3=x^6-6x^4+12x^2-8$

3. $(2x-\sqrt{y})(2x+\sqrt{y})=(2x)^2-(\sqrt{y})^2=4x^2-y$

4. $(x+y-1)(x+y+1)=[(x+y)-1][(x+y)+1]=(x+y)^2-1^2=x^2+2xy+y^2-1$

5. $(x+y+z)(x-y-z)=[x+(y+z)][x-(y+z)]=x^2-(y+z)^2=x^2-y^2-2yz-z^2$

6. $(2x-3+y)(2x+3+y)=[(2x+y)-3][(2x+y)+3]=(2x+y)^2-3^2=4x^2+4xy+y^2-9$

Factoring Common Factors

EXAMPLES:

1. $3x^2 - 6x = 3x(x-2)$

2. $8x^4y^2 + 6x^3y^3 - 2xy^4 = (2xy^2)(4x^3) + (2xy^2)(3x^2y) + (2xy^2)(-y^2) = 2xy^2(4x^3 + 3x^2y - y^2)$

Factoring Trinomials

To factor a trinomial of the form $x^2 + bx + c$, we note that

$$(x+r)(x+s) = x^2 + (r+s)x + rs$$

so we need to choose numbers r and s so that $r + s = b$ and $rs = c$.

EXAMPLES: We have

$$x^2 + 7x + 12 = (x+3)(x+4) \qquad\qquad x^2 - 2x - 3 = (x-3)(x+1)$$

To factor a trinomial of the form $ax^2 + bx + c$ with $a \neq 1$, we look for factors of the form $px + r$ and $qx + s$:

$$ax^2 + bx + c = (px+r)(qx+s) = pqx^2 + (ps+qr)x + rs$$

Therefore, we try to find numbers p, q, r, and s such that

$$pq = a, \quad rs = c, \quad ps + qr = b.$$

If these numbers are all integers, then we will have a limited number of possibilities to try for p, q, r, and s.

EXAMPLE: To factor $6x^2 + 7x - 5$, we note that we can factor 6 as $6 \cdot 1$ or $3 \cdot 2$, and -5 as $-5 \cdot 1$ or $5 \cdot (-1)$. By trying these possibilities, we arrive at the factorization

$$6x^2 + 7x - 5 = (3x+5)(2x-1)$$

Here is an other way to get the same factorization:

$$6x^2 + 7x - 5 = \left[\begin{array}{r} 6 \cdot (-5) = -30 = 10 \cdot (-3) \\ 7 = 10 + (-3) \end{array}\right] = 6x^2 - 3x + 10x - 5 = 3x(2x-1) + 5(2x-1)$$

$$= (2x-1)(3x+5)$$

EXAMPLE: We have

$$8x^2 - 14x + 3 = \left[\begin{array}{r} 8 \cdot 3 = 24 = (-12) \cdot (-2) \\ -14 = (-12) + (-2) \end{array}\right] = 8x^2 - 12x - 2x + 3 = 4x(2x-3) - (2x-3)$$

$$= 4x(2x-3) - 1(2x-3)$$

$$= (2x-3)(4x-1)$$

EXAMPLE: We have

$$(5a+1)^2 - 2(5a+1) - 3 = [\text{Put } y = 5a+1] = y^2 - 2y - 3 = (y-3)(y+1)$$

$$= [(5a+1) - 3][(5a+1) + 1]$$

$$= (5a-2)(5a+2)$$

Special Factoring Formulas

FACTORING FORMULAS

Formula	Name
1. $A^2 - B^2 = (A - B)(A + B)$	Difference of squares
2. $A^2 + 2AB + B^2 = (A + B)^2$	Perfect square
3. $A^2 - 2AB + B^2 = (A - B)^2$	Perfect square
4. $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$	Difference of cubes
5. $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$	Sum of cubes

EXAMPLES:

1. $4x^2 - 25 = (2x)^2 - 5^2 = (2x - 5)(2x + 5)$
2. $(x + y)^2 - z^2 = (x + y - z)(x + y + z)$
3. $27x^3 - 1 = (3x)^3 - 1^3 = (3x - 1)[(3x)^2 + (3x)(1) + 1^2] = (3x - 1)(9x^2 + 3x + 1)$
4. $x^6 + 8 = (x^2)^3 + 2^3 = (x^2 + 2)(x^4 - 2x^2 + 4)$
5. $x^2 + 6x + 9 = x^2 + 2(x)(3) + 3^2 = (x + 3)^2$
6. $4x^2 - 4xy + y^2 = (2x)^2 - 2(2x)(y) + y^2 = (2x - y)^2$

Factoring an Expression Completely

EXAMPLES:

1. $x^4 - y^4 = (x^2)^2 - (y^2)^2 = (x^2 - y^2)(x^2 + y^2) = (x - y)(x + y)(x^2 + y^2)$
2. $2x^4 - 8x^2 = 2x^2(x^2 - 4) = 2x^2(x - 2)(x + 2)$
3. $x^5y^2 - xy^6 = xy^2(x^4 - y^4) = xy^2(x^2 + y^2)(x^2 - y^2) = xy^2(x^2 + y^2)(x + y)(x - y)$
4. $3x^{3/2} - 9x^{1/2} + 6x^{-1/2} = 3x^{-1/2}(x^2 - 3x + 2) = 3x^{-1/2}(x - 1)(x - 2)$
5. $$(2 + x)^{-2/3}x + (2 + x)^{1/3} = (2 + x)^{-2/3}[x + (2 + x)] = (2 + x)^{-2/3}(2 + 2x) = 2(2 + x)^{-2/3}(1 + x)$$

Factoring by Grouping Terms

EXAMPLES:

1. $x^3 + x^2 + 4x + 4 = (x^3 + x^2) + (4x + 4) = x^2(x + 1) + 4(x + 1) = (x^2 + 4)(x + 1)$
2. $x^3 - 2x^2 - 3x + 6 = (x^3 - 2x^2) - (3x - 6) = x^2(x - 2) - 3(x - 2) = (x^2 - 3)(x - 2)$