Section 1.3 Algebraic Expressions
Adding and Subtracting Polynomials

**POLYNOMIALS**

A polynomial in the variable $x$ is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are real numbers and $n$ is a nonnegative integer. If $a_n \neq 0$, then the polynomial has degree $n$. The monomials $a_k x^k$ that make up the polynomial are called the terms of the polynomial.

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Type</th>
<th>Terms</th>
<th>Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2x^2 - 3x + 4$</td>
<td>trinomial</td>
<td>$2x^2, -3x, 4$</td>
<td>2</td>
</tr>
<tr>
<td>$x^8 + 5x$</td>
<td>binomial</td>
<td>$x^8, 5x$</td>
<td>8</td>
</tr>
<tr>
<td>$8 - x + x^2 - \frac{1}{2}x^3$</td>
<td>four terms</td>
<td>$-\frac{1}{2}x^3, x^2, -x, 8$</td>
<td>3</td>
</tr>
<tr>
<td>$5x + 1$</td>
<td>binomial</td>
<td>$5x, 1$</td>
<td>1</td>
</tr>
<tr>
<td>$9x^5$</td>
<td>monomial</td>
<td>$9x^5$</td>
<td>5</td>
</tr>
<tr>
<td>$6$</td>
<td>monomial</td>
<td>$6$</td>
<td>0</td>
</tr>
</tbody>
</table>

**EXAMPLES:**

1. \((x^3 - 6x^2 + 2x + 4) + (x^3 + 5x^2 - 7x) = (x^3 + x^3) + (-6x^2 + 5x^2) + (2x - 7x) + 4\\= 2x^3 - x^2 - 5x + 4\\

2. \((x^3 - 6x^2 + 2x + 4) - (x^3 + 5x^2 - 7x) = x^3 - 6x^2 + 2x + 4 - x^3 - 5x^2 + 7x\\= (x^3 - x^3) + (-6x^2 - 5x^2) + (2x + 7x) + 4\\= -11x^2 + 9x + 4\\

3. \(x^3 - 6x^2 + 2x + 4 - x^3 + 5x^2 - 7x = (x^3 - x^3) + (-6x^2 + 5x^2) + (2x - 7x) + 4\\= -x^2 - 5x + 4\\

4. \(5(3t - 4) - (t^2 + 2) - 2t(t - 3) = 15t - 20 - t^2 - 2 - 2t^2 + 6t\\= (-t^2 - 2t^2) + (15t + 6t) - 20 - 2\\= -3t^2 + 21t - 22\\
Multiplying Algebraic Expressions

EXAMPLES:

1. \((2x + 1)(3x - 5) = 2x(3x - 5) + 1(3x - 5)\)
   \[= (2x \cdot 3x - 2x \cdot 5) + (1 \cdot 3x - 1 \cdot 5)\]
   \[= (6x^2 - 10x) + (3x - 5)\]
   \[= 6x^2 - 10x + 3x - 5\]
   \[= 6x^2 - 7x - 5\]

2. \((2x - 3)(x^2 - 5x + 4) = 2x(x^2 - 5x + 4) - 3(x^2 - 5x + 4)\)
   \[= (2x \cdot x^2 - 2x \cdot 5x + 2x \cdot 4) - (3 \cdot x^2 - 3 \cdot 5x + 3 \cdot 4)\]
   \[= (2x^3 - 10x^2 + 8x) - (3x^2 - 15x + 12)\]
   \[= 2x^3 - 10x^2 + 8x - 3x^2 + 15x - 12\]
   \[= 2x^3 - 13x^2 + 23x - 12\]

Special Product Formulas

SPECIAL PRODUCT FORMULAS

If \(A\) and \(B\) are any real numbers or algebraic expressions, then

1. \((A + B)(A - B) = A^2 - B^2\) \quad Sum and difference of same terms
2. \((A + B)^2 = A^2 + 2AB + B^2\) \quad Square of a sum
3. \((A - B)^2 = A^2 - 2AB + B^2\) \quad Square of a difference
4. \((A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3\) \quad Cube of a sum
5. \((A - B)^3 = A^3 - 3A^2B + 3AB^2 - B^3\) \quad Cube of a difference

EXAMPLES:

1. \((3x + 5)^2 = (3x)^2 + 2(3x)(5) + 5^2 = 9x^2 + 30x + 25\)

2. \((x^2 - 2)^3 = (x^2)^3 - 3(x^2)^2(2) + 3(x^2)(2^2) - 2^3 = x^6 - 6x^4 + 12x^2 - 8\)

3. \((2x - \sqrt{y})(2x + \sqrt{y}) = (2x)^2 - (\sqrt{y})^2 = 4x^2 - y\)

4. \((x + y - 1)(x + y + 1) = [(x + y) - 1][(x + y) + 1] = (x + y)^2 - 1^2 = x^2 + 2xy + y^2 - 1\)

5. \((x + y + z)(x - y - z) = [x + (y + z)](x - (y + z)) = x^2 - (y + z)^2 = x^2 - y^2 - 2yz - z^2\)

6. \((2x - 3 + y)(2x + 3 + y) = [(2x + y) - 3][(2x + y) + 3] = (2x + y)^2 - 3^2 = 4x^2 + 4xy + y^2 - 9\)
Factoring Common Factors

EXAMPLES:
1. \(3x^2 - 6x = 3x(x - 2)\)
2. \(8x^4y^2 + 6x^3y^3 - 2xy^4 = (2xy^2)(4x^3) + (2xy^2)(3x^2y) + (2xy^2)(-y^2) = 2xy^2(4x^3 + 3x^2y - y^2)\)

Factoring Trinomials

To factor a trinomial of the form \(x^2 + bx + c\), we note that
\[(x + r)(x + s) = x^2 + (r + s)x + rs\]
so we need to choose numbers \(r\) and \(s\) so that \(r + s = b\) and \(rs = c\).

EXAMPLES: We have
\[x^2 + 7x + 12 = (x + 3)(x + 4)\]
\[x^2 - 2x - 3 = (x - 3)(x + 1)\]
To factor a trinomial of the form \(ax^2 + bx + c\) with \(a \neq 1\), we look for factors of the form \(px + r\) and \(qx + s\):
\[ax^2 + bx + c = (px + r)(qx + s) = pqx^2 + (ps + qr)x + rs\]
Therefore, we try to find numbers \(p, q, r,\) and \(s\) such that
\[pq = a, \quad rs = c, \quad ps + qr = b.\]
If these numbers are all integers, then we will have a limited number of possibilities to try for \(p, q, r,\) and \(s\).

EXAMPLE: To factor \(6x^2 + 7x - 5\), we note that we can factor 6 as \(6 \cdot 1\) or \(3 \cdot 2\), and \(-5\) as \(-5 \cdot 1\) or \(5 \cdot (-1)\). By trying these possibilities, we arrive at the factorization
\[6x^2 + 7x - 5 = (3x + 5)(2x - 1)\]
Here is another way to get the same factorization:
\[6x^2 + 7x - 5 = \left[6 \cdot (-5) = -30 = 10 \cdot (-3) \right] = 6x^2 - 3x + 10x - 5 = 3x(2x - 1) + 5(2x - 1) = (2x - 1)(3x + 5)\]

EXAMPLE: We have
\[8x^2 - 14x + 3 = \left[8 \cdot 3 = 24 = (-12) \cdot (-2) \right] = 8x^2 - 12x - 2x + 3 = 4x(2x - 3) - (2x - 3) = 4x(2x - 3) - 1(2x - 3) = (2x - 3)(4x - 1)\]

EXAMPLE: We have
\[(5a + 1)^2 - 2(5a + 1) - 3 = \text{[Put } y = 5a + 1\text{]} = y^2 - 2y - 3 = (y - 3)(y + 1) = [(5a + 1) - 3][(5a + 1) + 1] = (5a - 2)(5a + 2)\]
Special Factoring Formulas

### FACTORING FORMULAS

<table>
<thead>
<tr>
<th>Formula</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>2. $A^2 + 2AB + B^2 = (A + B)^2$</td>
<td>Perfect square</td>
</tr>
<tr>
<td>3. $A^2 - 2AB + B^2 = (A - B)^2$</td>
<td>Perfect square</td>
</tr>
<tr>
<td>5. $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$</td>
<td>Sum of cubes</td>
</tr>
</tbody>
</table>

### EXAMPLES:

1. $4x^2 - 25 = (2x)^2 - 5^2 = (2x - 5)(2x + 5)$
2. $(x + y)^2 - z^2 = (x + y - z)(x + y + z)$
3. $27x^3 - 1 = (3x)^3 - 1^3 = (3x - 1)(9x^2 + 3x + 1)$
4. $x^6 + 8 = (x^2)^3 + 2^3 = (x^2 + 2)(x^4 - 2x^2 + 4)$
5. $x^2 + 6x + 9 = x^2 + 2(x)(3) + 3^2 = (x + 3)^2$
6. $4x^2 - 4xy + y^2 = (2x)^2 - 2(2x)(y) + y^2 = (2x - y)^2$

### Factoring an Expression Completely

EXAMPLES:

1. $x^4 - y^4 = (x^2)^2 - (y^2)^2 = (x^2 - y^2)(x^2 + y^2) = (x - y)(x + y)(x^2 + y^2)$
2. $2x^4 - 8x^2 = 2x^2(x^2 - 4) = 2x^2(x - 2)(x + 2)$
3. $x^5 y^2 - xy^6 = xy^2(x^4 - y^4) = xy^2(x^2 + y^2)(x^2 - y^2) = xy^2(x^2 + y^2)(x + y)(x - y)$
4. $3x^{3/2} - 9x^{1/2} + 6x^{-1/2} = 3x^{-1/2}(x^2 - 3x + 2) = 3x^{-1/2}(x - 1)(x - 2)$
5. $(2 + x)^{-2/3}x + (2 + x)^{1/3} = (2 + x)^{-2/3}[x + (2 + x)] = (2 + x)^{-2/3}(2 + 2x)$

   \[
   = 2(2 + x)^{-2/3}(1 + x)
   \]

### Factoring by Grouping Terms

EXAMPLES:

1. $x^3 + x^2 + 4x + 4 = (x^3 + x^2) + (4x + 4) = x^2(x + 1) + 4(x + 1) = (x^2 + 4)(x + 1)$
2. $x^3 - 2x^2 - 3x + 6 = (x^3 - 2x^2) - (3x - 6) = x^2(x - 2) - 3(x - 2) = (x^2 - 3)(x - 2)$