Section 1.2 Exponents and Radicals

Integer Exponents

A product of identical numbers is usually written in exponential notation. For example, $5 \cdot 5 \cdot 5$ is written as 5^3. In general, we have the following definition.

EXPONENTIAL NOTATION

If a is any real number and n is a positive integer, then the ***n*th power** of a is

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

The number a is called the **base**, and n is called the **exponent**.

EXAMPLES:

(a) $\left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{32}$

(b) $(-3)^4 = (-3) \cdot (-3) \cdot (-3) \cdot (-3) = 81$

(c) $-3^4 = -(3 \cdot 3 \cdot 3 \cdot 3) = -81$

ZERO AND NEGATIVE EXPONENTS

If $a \neq 0$ is any real number and n is a positive integer, then

$$a^0 = 1 \qquad \text{and} \qquad a^{-n} = \frac{1}{a^n}$$

EXAMPLES:

(a) $\left(\frac{4}{7}\right)^0 = 1, \quad \left(\pi + \sqrt{\frac{a^3+b}{c^2+d^4+2}}\right)^0 = 1$

(b) 0^0 is undefined

(c) $x^{-1} = \frac{1}{x^1} = \frac{1}{x}$

(d) $(-2)^{-3} = \frac{1}{(-2)^3} = \frac{1}{-8} = -\frac{1}{8}$

(e) $-2^{-3} = -\frac{1}{2^3} = -\frac{1}{8}$

Rules for Working with Exponents

In the table the bases a and b are real numbers, and the exponents m and n are integers.

LAWS OF EXPONENTS

Law	Example	Description
1. $a^m a^n = a^{m+n}$	$3^2 \cdot 3^5 = 3^{2+5} = 3^7$	To multiply two powers of the same number, add the exponents.
2. $\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{3^5}{3^2} = 3^{5-2} = 3^3$	To divide two powers of the same number, subtract the exponents.
3. $(a^m)^n = a^{mn}$	$(3^2)^5 = 3^{2 \cdot 5} = 3^{10}$	To raise a power to a new power, multiply the exponents.
4. $(ab)^n = a^n b^n$	$(3 \cdot 4)^2 = 3^2 \cdot 4^2$	To raise a product to a power, raise each factor to the power.
5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$	$\left(\dfrac{3}{4}\right)^2 = \dfrac{3^2}{4^2}$	To raise a quotient to a power, raise both numerator and denominator to the power.

EXAMPLES:

(a) $x^4 x^7 \overset{(1)}{=} x^{4+7} = x^{11}$

(b) $y^4 y^{-7} \overset{(1)}{=} y^{4+(-7)} = y^{-3} = \dfrac{1}{y^3}$

(c) $\dfrac{c^9}{c^5} \overset{(2)}{=} c^{9-5} = c^4$

(d) $(b^{-4})^{-5} \overset{(3)}{=} b^{(-4)\cdot(-5)} = b^{20}$

(e) $(3x)^3 \overset{(4)}{=} 3^3 x^3 = 27x^3$

(f) $\left(\dfrac{x}{2}\right)^5 \overset{(5)}{=} \dfrac{x^5}{2^5} = \dfrac{x^5}{32}$

EXAMPLES: Simplify

(a) $(2a^3b^2)(3ab^4)^3$ (b) $\left(\dfrac{x}{y}\right)^3 \left(\dfrac{y^2x}{z}\right)^4$

Solution:

(a) $(2a^3b^2)(3ab^4)^3 \overset{(4)}{=} (2a^3b^2)[3^3a^3(b^4)^3] \overset{(3)}{=} (2a^3b^2)(27a^3b^{12}) = (2)(27)a^3a^3b^2b^{12} \overset{(1)}{=} 54a^6b^{14}$

(b) $\left(\dfrac{x}{y}\right)^3 \left(\dfrac{y^2x}{z}\right)^4 \overset{(5),(4)}{=} \dfrac{x^3}{y^3} \cdot \dfrac{(y^2)^4x^4}{z^4} \overset{(3)}{=} \dfrac{x^3}{y^3} \cdot \dfrac{y^8x^4}{z^4} = (x^3x^4)\left(\dfrac{y^8}{y^3}\right)\dfrac{1}{z^4} \overset{(1),(2)}{=} \dfrac{x^7y^5}{z^4}$

EXAMPLE: Simplify $\left(\dfrac{x^4z^7}{4y^5}\right)\left(\dfrac{2x^3y^3}{z^3}\right)^2$.

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Solution:

$$\left(\frac{x^4z^7}{4y^5}\right)\left(\frac{2x^3y^3}{z^3}\right)^2 \overset{(5),(4)}{=} \frac{x^4z^7}{4y^5}\cdot\frac{2^2(x^3)^2(y^3)^2}{(z^3)^2} \overset{(3)}{=} \frac{x^4z^7}{4y^5}\cdot\frac{4x^6y^6}{z^6} = (x^4x^6)\left(\frac{y^6}{y^5}\right)\left(\frac{z^7}{z^6}\right) \overset{(1),(2)}{=} x^{10}yz$$

REMARK: Note that

$$(2x^3y^3)^2 = 2^2(x^3)^2(y^3)^2$$

but

$$(2+x^3+y^3)^2 \neq 2^2+(x^3)^2+(y^3)^2$$

LAWS OF EXPONENTS

Law	Example	Description
6. $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$	$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$	To raise a fraction to a negative power, invert the fraction and change the sign of the exponent.
7. $\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}$	$\frac{3^{-2}}{4^{-5}} = \frac{4^5}{3^2}$	To move a number raised to a power from numerator to denominator or from denominator to numerator, change the sign of the exponent.

EXAMPLES: Eliminate negative exponents and simplify each expression.

(a) $\frac{6st^{-4}}{2s^{-2}t^2}$ (b) $\left(\frac{y}{3z^3}\right)^{-2}$

Solution:

(a) $\frac{6st^{-4}}{2s^{-2}t^2} \overset{(7)}{=} \frac{6ss^2}{2t^2t^4} \overset{(1)}{=} \frac{3s^3}{t^6}$ or $\frac{6st^{-4}}{2s^{-2}t^2} \overset{(2)}{=} 3s^{1-(-2)}t^{-4-2} = 3s^3t^{-6} = \frac{3s^3}{t^6}$

(b) $\left(\frac{y}{3z^3}\right)^{-2} \overset{(6)}{=} \left(\frac{3z^3}{y}\right)^2 \overset{(5),(4)}{=} \frac{9z^6}{y^2}$

EXAMPLE: Eliminate negative exponents and simplify $\left(\frac{q^{-1}r^{-1}s^{-2}}{q^{-8}r^{-5}s}\right)^{-1}$.

EXAMPLE: Eliminate negative exponents and simplify $\left(\frac{q^{-1}r^{-1}s^{-2}}{q^{-8}r^{-5}s}\right)^{-1}$.

Solution 1: We have

$$\left(\frac{q^{-1}r^{-1}s^{-2}}{q^{-8}r^{-5}s}\right)^{-1} \overset{(6)}{=} \frac{q^{-8}r^{-5}s}{q^{-1}r^{-1}s^{-2}} \overset{(7)}{=} \frac{qrss^2}{q^8r^5} \overset{(1)}{=} \frac{qrs^3}{q^8r^5} \overset{(1),(7)}{=} \frac{s^3}{q^7r^4}$$

or

$$\left(\frac{q^{-1}r^{-1}s^{-2}}{q^{-8}r^{-5}s}\right)^{-1} \overset{(6)}{=} \frac{q^{-8}r^{-5}s}{q^{-1}r^{-1}s^{-2}} \overset{(2)}{=} q^{-8-(-1)}r^{-5-(-1)}s^{1-(-2)} = q^{-7}r^{-4}s^3 = \frac{s^3}{q^7r^4}$$

Solution 2: We have

$$\left(\frac{q^{-1}r^{-1}s^{-2}}{q^{-8}r^{-5}s}\right)^{-1} \overset{(3),(4),(5)}{=} \frac{qrs^2}{q^8r^5s^{-1}} \overset{(7)}{=} \frac{qrss^2}{q^8r^5} \overset{(1)}{=} \frac{qrs^3}{q^8r^5} \overset{(1),(7)}{=} \frac{s^3}{q^7r^4}$$

or

$$\left(\frac{q^{-1}r^{-1}s^{-2}}{q^{-8}r^{-5}s}\right)^{-1} \overset{(3),(4),(5)}{=} \frac{qrs^2}{q^8r^5s^{-1}} \overset{(2)}{=} q^{1-8}r^{1-5}s^{2-(-1)} = q^{-7}r^{-4}s^3 = \frac{s^3}{q^7r^4}$$

Scientific Notation

Exponential notation is used by scientists as a compact way of writing very large numbers and very small numbers.

SCIENTIFIC NOTATION

A positive number x is said to be written in **scientific notation** if it is expressed as follows:

$$x = a \times 10^n \qquad \text{where } 1 \leq a < 10 \text{ and } n \text{ is an integer}$$

EXAMPLES: $56,920 = 5.692 \times 10^4$, $9.3 \times 10^{-5} = 0.000093$.

Radicals

DEFINITION OF nth ROOT

If n is any positive integer, then the **principal nth root** of a is defined as follows:

$$\sqrt[n]{a} = b \qquad \text{means} \qquad b^n = a$$

If n is even, we must have $a \geq 0$ and $b \geq 0$.

PROPERTIES OF nth ROOTS

Property	Example
1. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$	$\sqrt[3]{-8 \cdot 27} = \sqrt[3]{-8}\sqrt[3]{27} = (-2)(3) = -6$
2. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt[4]{\dfrac{16}{81}} = \dfrac{\sqrt[4]{16}}{\sqrt[4]{81}} = \dfrac{2}{3}$
3. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt{\sqrt[3]{729}} = \sqrt[6]{729} = 3$
4. $\sqrt[n]{a^n} = a$ if n is odd	$\sqrt[3]{(-5)^3} = -5$, $\sqrt[5]{2^5} = 2$
5. $\sqrt[n]{a^n} = \lvert a \rvert$ if n is even	$\sqrt[4]{(-3)^4} = \lvert -3 \rvert = 3$

EXAMPLES:

1. $\sqrt[3]{x^4} = \sqrt[3]{x^3 x} = \sqrt[3]{x^3}\sqrt[3]{x} = x\sqrt[3]{x}$

2. $\sqrt[4]{81x^8y^4} = \sqrt[4]{81}\sqrt[4]{x^8}\sqrt[4]{y^4} = 3\sqrt[4]{(x^2)^4}|y| = 3x^2|y|$

3. $\sqrt{32} + \sqrt{200} = \sqrt{16 \cdot 2} + \sqrt{100 \cdot 2} = \sqrt{16}\sqrt{2} + \sqrt{100}\sqrt{2} = 4\sqrt{2} + 10\sqrt{2} = 14\sqrt{2}$

REMARK: Note that $\sqrt{32} + \sqrt{200} \neq \sqrt{32 + 200}$.

4. $\sqrt{25b} - \sqrt{b^3} = \sqrt{25b} - \sqrt{b^2 \cdot b} = \sqrt{25}\sqrt{b} - \sqrt{b^2}\sqrt{b} = 5\sqrt{b} - b\sqrt{b} = (5 - b)\sqrt{b}, \quad b \geq 0$

Rational Exponents

DEFINITION OF RATIONAL EXPONENTS

For any rational exponent m/n in lowest terms, where m and n are integers and $n > 0$, we define

$$a^{m/n} = (\sqrt[n]{a})^m \qquad \text{or, equivalently,} \qquad a^{m/n} = \sqrt[n]{a^m}$$

If n is even, then we require that $a \geq 0$.

REMARK 1: With this definition it can be proved that *the Laws of Exponents also hold for rational exponents.*

REMARK 2: It is important that $a \geq 0$ if n is even in the definition above. Otherwise contradictions are possible. For example,

$$-1 = (-1)^1 \stackrel{?}{=} (-1)^{2/2} \stackrel{???}{=} \sqrt{(-1)^2} = \sqrt{1} = 1$$

EXAMPLES:

1. $4^{1/2} = \sqrt{4} = 2$

2 $8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4 \quad \text{or} \quad 8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$

3. $(125)^{-1/3} = \dfrac{1}{125^{1/3}} = \dfrac{1}{\sqrt[3]{125}} = \dfrac{1}{5}$

4. $\dfrac{1}{\sqrt[3]{x^4}} = \dfrac{1}{x^{4/3}} = x^{-4/3}$

5. $a^{1/3}a^{7/3} = a^{1/3+7/3} = a^{8/3}$

6. $\dfrac{a^{2/5}a^{7/5}a^{-1/5}}{a^{3/5}a^{-4/5}} = a^{2/5+7/5+(-1/5)-3/5-(-4/5)} = a^{2/5+7/5-1/5-3/5+4/5} = a^{9/5}$

7. $(2a^3b^4)^{3/2} = 2^{3/2}(a^3)^{3/2}(b^4)^{3/2} = (\sqrt{2})^3 a^{3(3/2)}b^{4(3/2)} = 2\sqrt{2}a^{9/2}b^6$

8. $\left(\dfrac{2x^{3/4}}{y^{1/3}}\right)^3 \left(\dfrac{y^4}{x^{-1/2}}\right) = \dfrac{2^3(x^{3/4})^3}{(y^{1/3})^3} \cdot (y^4x^{1/2}) = \dfrac{8x^{9/4}}{y} \cdot y^4x^{1/2} = 8x^{11/4}y^3$

9. $(2\sqrt{x})(3\sqrt[3]{x}) = (2x^{1/2})(3x^{1/3}) = 6x^{1/2+1/3} = 6x^{1/2+1/3} = 6x^{5/6}$

10. $\sqrt{x\sqrt{x}} = (x \cdot x^{1/2})^{1/2} = (x^{3/2})^{1/2} = x^{3/4}$

Rationalizing the Denominator

EXAMPLES:

1. We have

$$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

2. We have

$$\frac{4}{\sqrt[3]{5}} = \frac{4\sqrt[3]{5^2}}{\sqrt[3]{5}\sqrt[3]{5^2}} = \frac{4\sqrt[3]{5^2}}{\sqrt[3]{5\cdot 5^2}} = \frac{4\sqrt[3]{5^2}}{\sqrt[3]{5^{1+2}}} = \frac{4\sqrt[3]{5^2}}{\sqrt[3]{5^3}} = \frac{4\sqrt[3]{25}}{5}$$

or, in short,

$$\frac{4}{\sqrt[3]{5}} = \frac{4\sqrt[3]{5^2}}{\sqrt[3]{5}\sqrt[3]{5^2}} = \frac{4\sqrt[3]{25}}{5}$$

3. We have

$$\frac{1}{\sqrt[5]{x^2}} = \frac{\sqrt[5]{x^3}}{\sqrt[5]{x^2}\sqrt[5]{x^3}} = \frac{\sqrt[5]{x^3}}{\sqrt[5]{x^2x^3}} = \frac{\sqrt[5]{x^3}}{\sqrt[5]{x^{2+3}}} = \frac{\sqrt[5]{x^3}}{\sqrt[5]{x^5}} = \frac{\sqrt[5]{x^3}}{x}$$

or, in short,

$$\frac{1}{\sqrt[5]{x^2}} = \frac{\sqrt[5]{x^3}}{\sqrt[5]{x^2}\sqrt[5]{x^3}} = \frac{\sqrt[5]{x^3}}{x}$$