Section 1.10 Lines

The Slope of a Line

**Slope of a Line**

The slope \( m \) of a nonvertical line that passes through the points \( A(x_1, y_1) \) and \( B(x_2, y_2) \) is

\[
m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}
\]

The slope of a vertical line is not defined.

EXAMPLE: Find the slope of the line that passes through the points \( P(2, 1) \) and \( Q(8, 5) \).

Solution: We have

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 1}{8 - 2} = \frac{4}{6} = \frac{2}{3}
\]
EXAMPLE: Find the slope of the line that passes through the points $P(-2, -1)$ and $Q(-8, 5)$.

Solution: We have

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - (-1)}{-8 - (-2)} = \frac{5 + 1}{-8 + 2} = \frac{6}{-6} = -1$$

EXAMPLE: Find the slope of the line that passes through the points $P(-3, 1)$ and $Q(5, 6)$.

Solution: We have

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 1}{5 - (-3)} = \frac{6 - 1}{5 + 3} = \frac{5}{8}$$

**Point-Slope Form of the Equation of a Line**

From the picture above it follows that

$$\frac{y - y_1}{x - x_1} = m$$

which can be rewritten as

$$y - y_1 = m(x - x_1)$$

**Point-Slope Form of the Equation of a Line**

An equation of the line that passes through the point $(x_1, y_1)$ and has slope $m$ is

$$y - y_1 = m(x - x_1)$$

EXAMPLE: Find an equation of the line through $(1, -3)$ with slope $-\frac{1}{2}$ and sketch the line.
EXAMPLE: Find an equation of the line through (1, −3) with slope \(-\frac{1}{2}\) and sketch the line.

Solution: Using the point-slope form with \(m = -\frac{1}{2}\), \(x_1 = 1\), and \(y_1 = -3\), we obtain an equation of the line as

\[
\begin{align*}
y - (-3) &= -\frac{1}{2}(x - 1) \\
y + 3 &= -\frac{1}{2}(x - 1) \\
2(y + 3) &= -(x - 1) \\
2y + 6 &= -x + 1 \\
2y + 6 + x - 1 &= 0 \\
x + 2y + 5 &= 0
\end{align*}
\]

EXAMPLE: Find an equation of the line through the points (−1, 2) and (3, −4).

Solution: The slope of the line is

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - 2}{3 - (-1)} = \frac{-6}{4} = -\frac{3}{2}
\]

Using the point-slope form with \(x_1 = -1\) and \(y_1 = 2\), we obtain

\[
\begin{align*}
y - 2 &= -\frac{3}{2}(x - (-1)) \\
y - 2 &= -\frac{3}{2}(x + 1) \\
2(y - 2) &= -3(x + 1) \\
2y - 4 &= -3x - 3 \\
2y - 4 + 3x + 3 &= 0 \\
3x + 2y - 1 &= 0
\end{align*}
\]

or

\[
\begin{align*}
y - (-4) &= -\frac{3}{2}(x - 3) \\
y + 4 &= -\frac{3}{2}(x - 3) \\
2(y + 4) &= -3(x - 3) \\
2y + 8 &= -3x + 9 \\
2y + 8 + 3x - 9 &= 0 \quad \text{or} \\
3x + 2y - 1 &= 0
\end{align*}
\]
Slope-Intercept Form of the Equation of a Line

An equation of the line that has slope $m$ and $y$-intercept $b$ is

$$y = mx + b$$

EXAMPLE: Find an equation of the line with slope 3 and $y$-intercept $-2$.
Solution: Since $m = 3$ and $b = -2$, from the slope-intercept form of the equation of a line we get

$$y = mx + b = 3x + (-2) = 3x - 2$$

EXAMPLE: Find an equation for the line that has $x$-intercept 6 and $y$-intercept 4.
Solution 1: Since the line passes through the points $(6, 0)$ and $(0, 4)$, the slope of this line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 0}{0 - 6} = \frac{4}{-6} = -\frac{2}{3}$$

Since the $y$-intercept is 4, it follows that $b = 4$. Therefore the slope-intercept equation of the line is

$$y = mx + b = -\frac{2}{3}x + 4$$

Solution 2: Since the $y$-intercept is 4, it follows that the slope-intercept equation of the line is $y = mx + 4$. Since the $x$-intercept is 6, the line passes through the point $(6, 0)$, therefore

$$0 = m \cdot 6 + 4 \implies -4 = 6m \implies m = \frac{-4}{6} = -\frac{2}{3}$$

and the same result follows.

EXAMPLE: Find the slope and $y$-intercept of the line $3y - 2x = 1$. 
EXAMPLE: Find the slope and \( y \)-intercept of the line \( 3y - 2x = 1 \).

Solution: We first write the equation in the form \( y = mx + b \):

\[
3y - 2x = 1 \\
3y = 2x + 1 \\
y = \frac{2x + 1}{3} = \frac{2}{3}x + \frac{1}{3}
\]

From the slope-intercept form of the equation of a line, we see that the slope is \( m = \frac{2}{3} \) and the \( y \)-intercept is \( b = \frac{1}{3} \).

EXAMPLE: Write the linear equation \( 2x - 3y = 15 \) in slope-intercept form, and sketch its graph. What are the slope and \( y \)-intercept?

Solution: We have

\[
2x - 3y = 15 \iff -3y = -2x + 15 \iff y = \frac{-2x + 15}{-3} = \frac{-2x}{-3} + \frac{15}{-3} = \frac{2}{3}x - 5
\]

Therefore the slope is \( \frac{2}{3} \) and \( y \)-intercept is \(-5\).

Vertical and Horizontal Lines

EXAMPLE:

(a) The graph of the equation \( x = 3 \) is a vertical line with \( x \)-intercept 3.

(b) The graph of the equation \( y = -2 \) is a horizontal line with \( y \)-intercept \(-2\).
General Equation of a Line

The graph of every **linear equation**

\[
Ax + By + C = 0 \quad (A, B \text{ not both zero})
\]

is a line. Conversely, every line is the graph of a linear equation.

**EXAMPLE:** Sketch the graph of the equation \(2x - 3y - 12 = 0\).

Solution 1: Since the equation is linear, its graph is a line. To draw the graph, it is enough to find any two points on the line. The intercepts are the easiest points to find.

- **x-intercept:** Substitute \(y = 0\), to get \(2x - 12 = 0 \implies 2x = 12 \implies x = \frac{12}{2} = 6\)

- **y-intercept:** Substitute \(x = 0\), to get \(-3y - 12 = 0 \implies -3y = 12 \implies y = \frac{12}{-3} = -4\)

With these points we can sketch the graph in the Figure below (left).

Solution 2: We write the equation in slope-intercept form:

\[
2x - 3y - 12 = 0
\]

\[
-3y = -2x + 12
\]

\[
y = \frac{-2x + 12}{-3} = \frac{-2x}{-3} + \frac{12}{-3} = \frac{2}{3}x - 4
\]

This equation is in the form \(y = mx + b\), so the slope is \(m = \frac{2}{3}\) and the \(y\)-intercept is \(b = -4\). To sketch the graph, we plot the \(y\)-intercept, and then move \(3\) units to the right and \(2\) units up as shown in the Figure above (right).
EXAMPLE: Find an equation of the line through the point \((5, 2)\) that is parallel to the line \(4x + 6y + 5 = 0\).

Solution: We have

\[
4x + 6y + 5 = 0 \\
6y = -4x - 5 \\
y = \frac{-4x - 5}{6} = \frac{-4x}{6} - \frac{5}{6} = -\frac{2}{3}x - \frac{5}{6}
\]

So the line has slope \(m = -\frac{2}{3}\). Since the required line is parallel to the given line, it also has slope \(m = -\frac{2}{3}\). From the point-slope form of the equation of a line, we get

\[
y - 2 = -\frac{2}{3}(x - 5) \\
3(y - 2) = -2(x - 5) \\
3y - 6 = -2x + 10 \\
3y - 6 + 2x - 10 = 0 \\
2x + 3y - 16 = 0
\]

Thus, the equation of the required line is \(2x + 3y - 16 = 0\).

EXAMPLE: Find an equation for the line that passes through the point \((3, -6)\) and is parallel to the line \(3x + y - 10 = 0\). 

\[
\text{Thus, the equation of the required line is } 2x + 3y - 16 = 0.
\]
EXAMPLE: Find an equation for the line that passes through the point \((3, -6)\) and is parallel to the line \(3x + y - 10 = 0\).

Solution: We have

\[ 3x + y - 10 = 0 \implies y = -3x + 10 \]

Therefore the slope of the line is \(-3\). It follows that an equation for the line that passes through the point \((3, -6)\) and is parallel to the line \(3x + y - 10 = 0\) is

\[ y - (-6) = -3(x - 3) \implies y + 6 = -3x + 9 \implies y = -3x + 3 \]

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**Perpendicular Lines**

Two lines with slopes \(m_1\) and \(m_2\) are perpendicular if and only if \(m_1m_2 = -1\), that is, their slopes are negative reciprocals:

\[ m_2 = -\frac{1}{m_1} \]

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EXAMPLE: Show that the points \(P(3, 3)\), \(Q(8, 17)\), and \(R(11, 5)\) are the vertices of a right triangle.

Solution: The slopes of the lines containing \(PR\) and \(QR\) are, respectively,

\[ m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 3}{11 - 3} = \frac{2}{8} = \frac{1}{4} \quad \text{and} \quad m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 17}{11 - 8} = \frac{-12}{3} = -4 \]

Since \(m_1m_2 = -1\), these lines are perpendicular and so \(PQR\) is a right triangle. It is sketched in the Figure above (right).

EXAMPLE: Find an equation of the line that is perpendicular to the line \(3x - 2y + 7 = 0\) and passes through the origin.
EXAMPLE: Find an equation of the line that is perpendicular to the line $3x - 2y + 7 = 0$ and passes through the origin.

Solution: We first find the slope of the line $3x - 2y + 7 = 0$:

$$3x - 2y + 7 = 0$$
$$-2y = -3x - 7$$
$$y = \frac{-3x - 7}{-2} = \frac{-3x}{-2} - \frac{7}{-2} = \frac{3}{2}x + \frac{7}{2}$$

So the line has slope $m = \frac{3}{2}$. Thus, the slope of a perpendicular line is the negative reciprocal, that is, $-\frac{2}{3}$. Since the required line passes through $(0, 0)$, the point-slope form gives

$$y - 0 = -\frac{2}{3}(x - 0)$$
$$y = -\frac{2}{3}x$$

EXAMPLE: Let $P(-3, 1)$ and $Q(5, 6)$ be two points in the coordinate plane. Find the perpendicular bisector of the line that contains $P$ and $Q$.

Solution: The slope of the line that contains $P$ and $Q$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 6}{-3 - 5} = -\frac{5}{-8} = \frac{5}{8}$$

Therefore the slope of the perpendicular bisector of the line that contains $P$ and $Q$ is $-\frac{8}{5}$. We also need the midpoint of the segment $PQ$:

$$\left(\frac{-3 + 5}{2}, \frac{1 + 6}{2}\right) = \left(\frac{2}{2}, \frac{7}{2}\right) = \left(1, \frac{7}{2}\right)$$

It follows that an equation of the perpendicular bisector of the line that contains $P$ and $Q$ is

$$y - \frac{7}{2} = -\frac{8}{5}(x - 1)$$
$$y - \frac{7}{2} = -\frac{8}{5}x + \frac{8}{5}$$

so

$$y = -\frac{8}{5}x + \frac{8}{5} + \frac{7}{2} = -\frac{8}{5}x + \frac{8}{2 \cdot 5} + \frac{7 \cdot 5}{2 \cdot 5} = -\frac{8}{5}x + \frac{2 \cdot 8 + 7 \cdot 5}{2 \cdot 5} = -\frac{8}{5}x + \frac{16 + 35}{10} = -\frac{8}{5}x + \frac{51}{10}$$
Applications: Slope as Rate of Change

When a line is used to model the relationship between two quantities, the slope of the line is the rate of change of one quantity with respect to the other. For example, the graph in Figure (a) below gives the amount of gas in a tank that is being filled. The slope between the indicated points is

\[ m = \frac{6 \text{ gallons}}{3 \text{ minutes}} = 2 \text{ gal/min} \]

The slope is the rate at which the tank is being filled, 2 gallons per minute. In Figure (b) below, the tank is being drained at the rate of 0.03 gallon per minute, and the slope is $-0.03$.

EXAMPLE: A dam is built on a river to create a reservoir. The water level \( w \) in the reservoir is given by the equation

\[ w = 4.5t + 28 \]

where \( t \) is the number of years since the dam was constructed, and \( w \) is measured in feet.

(a) Sketch a graph of this equation.

(b) What do the slope and \( w \)-intercept of this graph represent?
EXAMPLE: A dam is built on a river to create a reservoir. The water level $w$ in the reservoir is given by the equation 

$$w = 4.5t + 28$$

where $t$ is the number of years since the dam was constructed, and $w$ is measured in feet.

(a) Sketch a graph of this equation.

(b) What do the slope and $w$-intercept of this graph represent?

Solution:

(a) This equation is linear, so its graph is a line. Since two points determine a line, we plot two points that lie on the graph and draw a line through them.

When $t = 0$, then $w = 4.5(0) + 28 = 28$, so $(0, 28)$ is on the line.

When $t = 2$, then $w = 4.5(2) + 28 = 37$, so $(2, 37)$ is on the line.

The line determined by these points is shown in the Figure below.

(b) The slope is $m = 4.5$; it represents the rate of change of water level with respect to time. This means that the water level increases 4.5 ft per year. The $w$-intercept is 28 and occurs when $t = 0$, so it represents the water level when the dam was constructed.