# Section 1.10 Lines

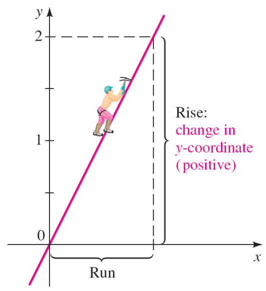
## The Slope of a Line

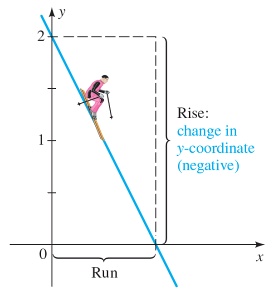
### Slope of a Line

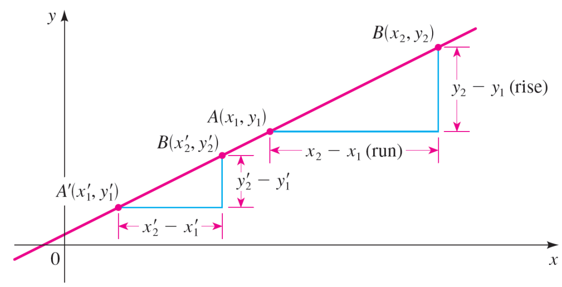
The **slope** $m$ of a nonvertical line that passes through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

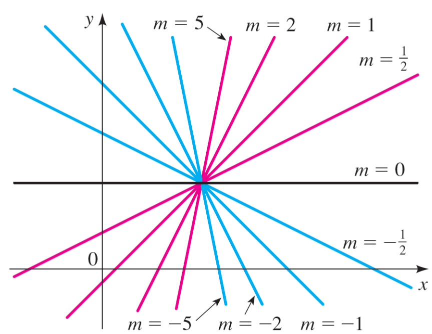
$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined.



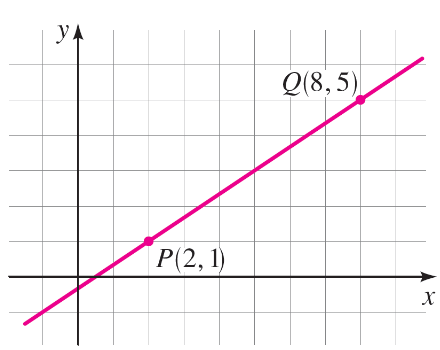




EXAMPLE: Find the slope of the line that passes through the points $P(2, 1)$ and $Q(8, 5)$.

Solution: We have

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 1}{8 - 2} = \frac{4}{6} = \frac{2}{3}$$



EXAMPLE: Find the slope of the line that passes through the points $P(-2,-1)$ and $Q(-8,5)$.

Solution: We have

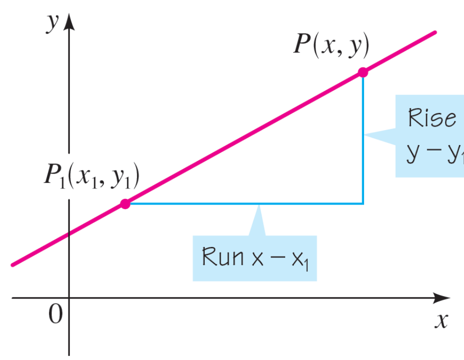
$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{5-(-1)}{-8-(-2)}=\frac{5+1}{-8+2}=\frac{6}{-6}=-1$$

EXAMPLE: Find the slope of the line that passes through the points $P(-3,1)$ and $Q(5,6)$.

Solution: We have

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{6-1}{5-(-3)}=\frac{6-1}{5+3}=\frac{5}{8}$$

## Point-Slope Form of the Equation of a Line




From the picture above it follows that

$$\frac{y-y_1}{x-x_1}=m$$

which can be rewritten as

$$y-y_1=m(x-x_1)$$

**Point-Slope Form of the Equation of a Line**

An equation of the line that passes through the point $(x_1, y_1)$ and has slope $m$ is

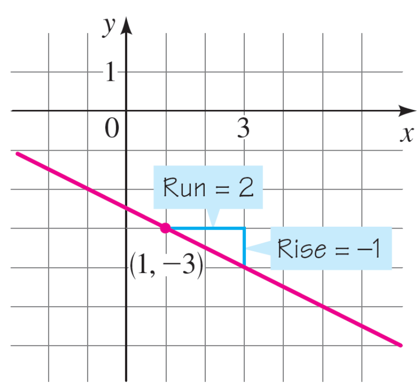
$$y-y_1=m(x-x_1)$$

EXAMPLE: Find an equation of the line through $(1,-3)$ with slope $-\dfrac{1}{2}$ and sketch the line.

EXAMPLE: Find an equation of the line through $(1, -3)$ with slope $-\dfrac{1}{2}$ and sketch the line.

Solution: Using the point-slope form with $m = -\dfrac{1}{2}$, $x_1 = 1$, and $y_1 = -3$, we obtain an equation of the line as

$$\begin{aligned} y - (-3) &= -\frac{1}{2}(x - 1) \\ y + 3 &= -\frac{1}{2}(x - 1) \\ 2(y + 3) &= -(x - 1) \\ 2y + 6 &= -x + 1 \\ 2y + 6 + x - 1 &= 0 \\ x + 2y + 5 &= 0 \end{aligned}$$




EXAMPLE: Find an equation of the line through the points $(-1, 2)$ and $(3, -4)$.

Solution: The slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - 2}{3 - (-1)} = \frac{-4 - 2}{3 + 1} = \frac{-6}{4} = -\frac{3}{2}$$

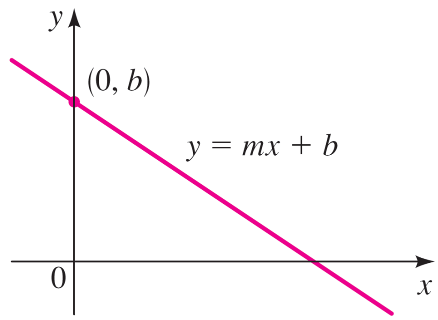
Using the point-slope form with $x_1 = -1$ and $y_1 = 2$, we obtain

$$\begin{aligned} y - 2 &= -\frac{3}{2}(x - (-1)) \\ y - 2 &= -\frac{3}{2}(x + 1) \\ 2(y - 2) &= -3(x + 1) \\ 2y - 4 &= -3x - 3 \\ 2y - 4 + 3x + 3 &= 0 \\ 3x + 2y - 1 &= 0 \end{aligned}$$

or

$$\begin{aligned} y - (-4) &= -\frac{3}{2}(x - 3) \\ y + 4 &= -\frac{3}{2}(x - 3) \\ 2(y + 4) &= -3(x - 3) \\ 2y + 8 &= -3x + 9 \\ 2y + 8 + 3x - 9 &= 0 \\ 3x + 2y - 1 &= 0 \end{aligned}$$

# Slope-Intercept Form of the Equation of a Line




From the picture above it follows that

$$y - b = m(x - 0)$$

which can be rewritten as

$$y = mx + b$$

## Slope-Intercept Form of the Equation of a Line

An equation of the line that has slope $m$ and $y$-intercept $b$ is

$$y = mx + b$$

EXAMPLE: Find an equation of the line with slope 3 and $y$-intercept $-2$.

Solution: Since $m = 3$ and $b = -2$, from the slope-intercept form of the equation of a line we get

$$y = mx + b = 3x + (-2) = 3x - 2$$

EXAMPLE: Find an equation for the line that has $x$-intercept 6 and $y$-intercept 4.

Solution 1: Since the line passes through the points $(6, 0)$ and $(0, 4)$, the slope of this line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 0}{0 - 6} = \frac{4}{-6} = -\frac{2}{3}$$

Since the $y$-intercept is 4, it follows that $b = 4$. Therefore the slope-intercept equation of the line is

$$y = mx + b = -\frac{2}{3}x + 4$$

Solution 2: Since the $y$-intercept is 4, it follows that the slope-intercept equation of the line is $y = mx + 4$. Since the $x$-intercept is 6, the line passes throught the point $(6, 0)$, therefore

$$0 = m \cdot 6 + 4 \quad \Longrightarrow \quad -4 = 6m \quad \Longrightarrow \quad m = \frac{-4}{6} = -\frac{2}{3}$$

and the same result follows.

EXAMPLE: Find the slope and $y$-intercept of the line $3y - 2x = 1$.

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Solution: We first write the equation in the form $y = mx + b$:

$$3y - 2x = 1$$

$$3y = 2x + 1$$

$$y = \frac{2x+1}{3} = \frac{2}{3}x + \frac{1}{3}$$

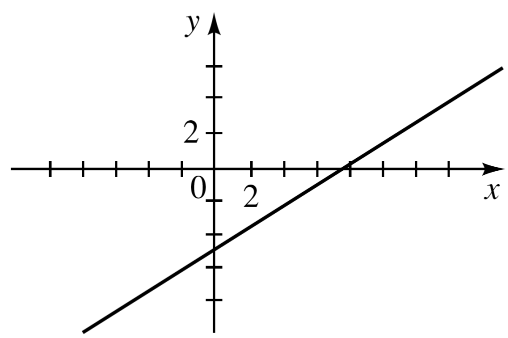
From the slope-intercept form of the equation of a line, we see that the slope is $m = \frac{2}{3}$ and the $y$-intercept is $b = \frac{1}{3}$.

EXAMPLE: Write the linear equation $2x - 3y = 15$ in slope-intercept form, and sketch its graph. What are the slope and $y$-intercept?

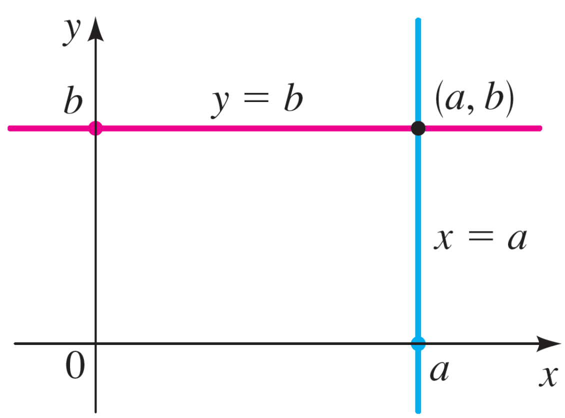
Solution: We have

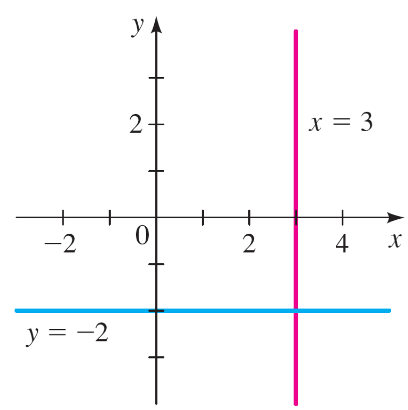
$$2x - 3y = 15 \implies -3y = -2x + 15 \implies y = \frac{-2x+15}{-3} = \frac{-2x}{-3} + \frac{15}{-3} = \frac{2}{3}x - 5$$

Therefore the slope is $\frac{2}{3}$ and $y$-intercept is $-5$.




## Vertical and Horizontal Lines




EXAMPLE:

(a) The graph of the equation $x = 3$ is a vertical line with $x$-intercept 3.

(b) The graph of the equation $y = -2$ is a horizontal line with $y$-intercept $-2$.

## General Equation of a Line

The graph of every **linear equation**

$$Ax + By + C = 0 \qquad (A, B \text{ not both zero})$$

is a line. Conversely, every line is the graph of a linear equation.

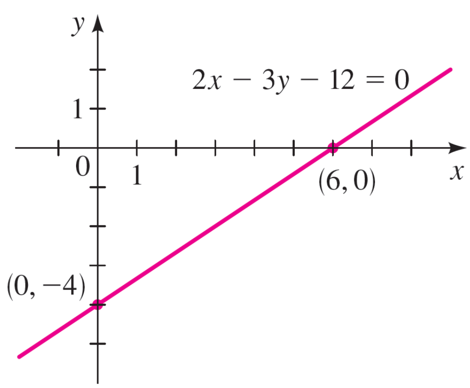
EXAMPLE: Sketch the graph of the equation $2x - 3y - 12 = 0$.

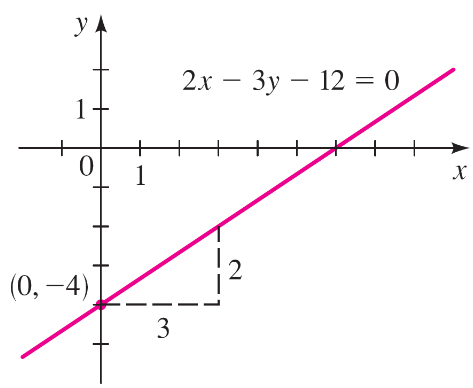
Solution 1: Since the equation is linear, its graph is a line. To draw the graph, it is enough to find any two points on the line. The intercepts are the easiest points to find.

$$x\text{-intercept: Substitute } y = 0, \text{ to get } 2x - 12 = 0 \implies 2x = 12 \implies x = \frac{12}{2} = 6$$

$$y\text{-intercept: Substitute } x = 0, \text{ to get } -3y - 12 = 0 \implies -3y = 12 \implies y = \frac{12}{-3} = -4$$

With these points we can sketch the graph in the Figure below (left).




Solution 2: We write the equation in slope-intercept form:

$$2x - 3y - 12 = 0$$

$$-3y = -2x + 12$$

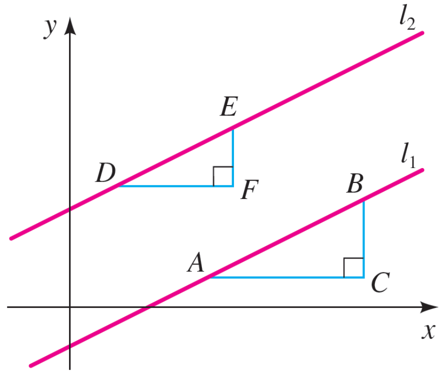
$$y = \frac{-2x + 12}{-3} = \frac{-2x}{-3} + \frac{12}{-3} = \frac{2}{3}x - 4$$

This equation is in the form $y = mx + b$, so the slope is $m = \dfrac{2}{3}$ and the $y$-intercept is $b = -4$. To sketch the graph, we plot the $y$-intercept, and then move 3 units to the right and 2 units up as shown in the Figure above (right).

## Parallel and Perpendicular Lines

### Parallel Lines

Two nonvertical lines are parallel if and only if they have the same slope.




EXAMPLE: Find an equation of the line through the point $(5, 2)$ that is parallel to the line $4x + 6y + 5 = 0$.

Solution: We have

$$4x + 6y + 5 = 0$$

$$6y = -4x - 5$$

$$y = \frac{-4x - 5}{6} = \frac{-4x}{6} - \frac{5}{6} = -\frac{2}{3}x - \frac{5}{6}$$

So the line has slope $m = -\frac{2}{3}$. Since the required line is parallel to the given line, it also has slope $m = -\frac{2}{3}$. From the point-slope form of the equation of a line, we get

$$y - 2 = -\frac{2}{3}(x - 5)$$

$$3(y - 2) = -2(x - 5)$$

$$3y - 6 = -2x + 10$$

$$3y - 6 + 2x - 10 = 0$$

$$2x + 3y - 16 = 0$$

Thus, the equation of the required line is $2x + 3y - 16 = 0$.

EXAMPLE: Find an equation for the line that passes through the point $(3, -6)$ and is parallel to the line $3x + y - 10 = 0$.

EXAMPLE: Find an equation for the line that passes through the point $(3, -6)$ and is parallel to the line $3x + y - 10 = 0$.

Solution: We have

$$3x + y - 10 = 0 \quad \Longrightarrow \quad y = -3x + 10$$

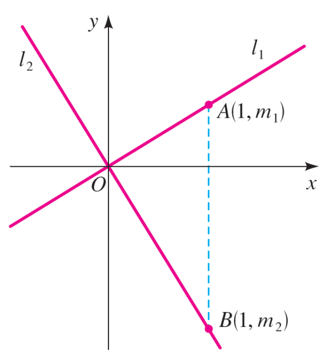
Therefore the slope of the line is $-3$. It follows that an equation for the line that passes through the point $(3, -6)$ and is parallel to the line $3x + y - 10 = 0$ is

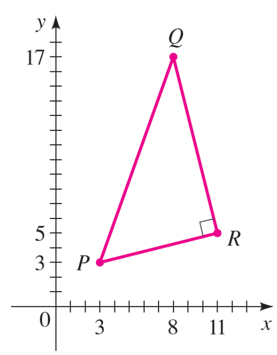
$$y - (-6) = -3(x - 3) \quad \Longrightarrow \quad y + 6 = -3x + 9 \quad \Longrightarrow \quad y = -3x + 3$$

## Perpendicular Lines

Two lines with slopes $m_1$ and $m_2$ are perpendicular if and only if $m_1 m_2 = -1$, that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}$$




EXAMPLE: Show that the points $P(3, 3)$, $Q(8, 17)$, and $R(11, 5)$ are the vertices of a right triangle.

Solution: The slopes of the lines containing $PR$ and $QR$ are, respectively,

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{11-3} = \frac{2}{8} = \frac{1}{4} \qquad \text{and} \qquad m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-17}{11-8} = \frac{-12}{3} = -4$$

Since $m_1 m_2 = -1$, these lines are perpendicular and so $PQR$ is a right triangle. It is sketched in the Figure above (right).

EXAMPLE: Find an equation of the line that is perpendicular to the line $3x - 2y + 7 = 0$ and passes through the origin.

EXAMPLE: Find an equation of the line that is perpendicular to the line $3x - 2y + 7 = 0$ and passes through the origin.

Solution: We first find the slope of the line $3x - 2y + 7 = 0$:

$$
\begin{aligned}
3x - 2y + 7 &= 0 \\
-2y &= -3x - 7 \\
y &= \frac{-3x - 7}{-2} = \frac{-3x}{-2} - \frac{7}{-2} = \frac{3}{2}x + \frac{7}{2}
\end{aligned}
$$

So the line has slope $m = \dfrac{3}{2}$. Thus, the slope of a perpendicular line is the negative reciprocal, that is, $-\dfrac{2}{3}$. Since the required line passes through $(0, 0)$, the point-slope form gives

$$
\begin{aligned}
y - 0 &= -\frac{2}{3}(x - 0) \\
y &= -\frac{2}{3}x
\end{aligned}
$$

EXAMPLE: Let $P(-3, 1)$ and $Q(5, 6)$ be two points in the coordinate plane. Find the perpendicular bisector of the line that contains $P$ and $Q$.

Solution: The slope of the line that contains $P$ and $Q$ is

$$
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 6}{-3 - 5} = \frac{-5}{-8} = \frac{5}{8}
$$

Therefore the slope of the perpendicular bisector of the line that contains $P$ and $Q$ is $-\dfrac{8}{5}$. We also need the midpoint of the segment $PQ$:

$$
\left(\frac{-3 + 5}{2}, \frac{1 + 6}{2}\right) = \left(\frac{2}{2}, \frac{7}{2}\right) = \left(1, \frac{7}{2}\right)
$$

It follows that an equation of the perpendicular bisector of the line that contains $P$ and $Q$ is

$$
\begin{aligned}
y - \frac{7}{2} &= -\frac{8}{5}(x - 1) \\
y - \frac{7}{2} &= -\frac{8}{5}x + \frac{8}{5}
\end{aligned}
$$

so

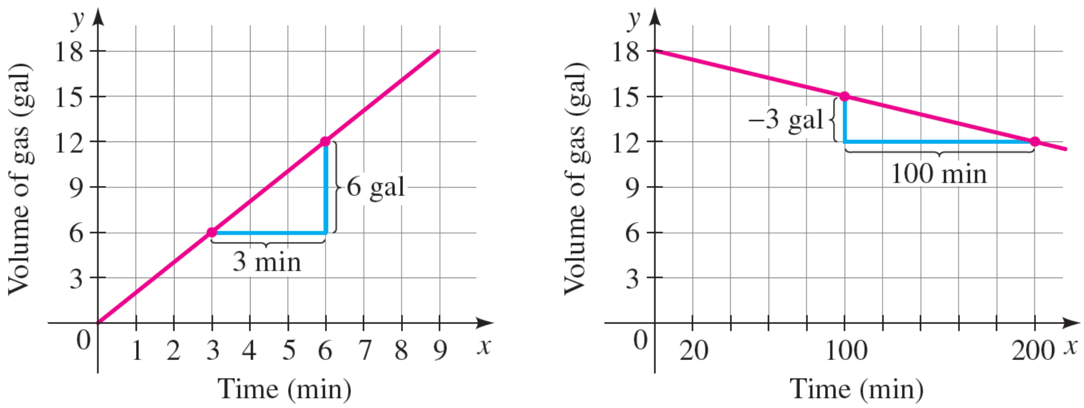
$$
y = -\frac{8}{5}x + \frac{8}{5} + \frac{7}{2} = -\frac{8}{5}x + \frac{2 \cdot 8}{2 \cdot 5} + \frac{7 \cdot 5}{2 \cdot 5} = -\frac{8}{5}x + \frac{2 \cdot 8 + 7 \cdot 5}{2 \cdot 5} = -\frac{8}{5}x + \frac{16 + 35}{10} = -\frac{8}{5}x + \frac{51}{10}
$$

# Applications: Slope as Rate of Change

When a line is used to model the relationship between two quantities, the slope of the line is the **rate of change** of one quantity with respect to the other. For example, the graph in Figure (a) below gives the amount of gas in a tank that is being filled. The slope between the indicated points is

$$m = \frac{6 \text{ gallons}}{3 \text{ minutes}} = 2 \text{ gal/min}$$

The slope is the *rate* at which the tank is being filled, 2 gallons per minute. In Figure (b) below, the tank is being drained at the *rate* of 0.03 gallon per minute, and the slope is $-0.03$.




(a) Tank filled at 2 gal/min
Slope of line is 2

(b) Tank drained at 0.03 gal/min
Slope of line is −0.03

EXAMPLE: A dam is built on a river to create a reservoir. The water level $w$ in the reservoir is given by the equation

$$w = 4.5t + 28$$

where $t$ is the number of years since the dam was constructed, and $w$ is measured in feet.

(a) Sketch a graph of this equation.

(b) What do the slope and $w$-intercept of this graph represent?

EXAMPLE: A dam is built on a river to create a reservoir. The water level $w$ in the reservoir is given by the equation

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where $t$ is the number of years since the dam was constructed, and $w$ is measured in feet.

(a) Sketch a graph of this equation.

(b) What do the slope and $w$-intercept of this graph represent?

Solution:

(a) This equation is linear, so its graph is a line. Since two points determine a line, we plot two points that lie on the graph and draw a line through them.

When $t = 0$, then $w = 4.5(0) + 28 = 28$, so $(0, 28)$ is on the line.

When $t = 2$, then $w = 4.5(2) + 28 = 37$, so $(2, 37)$ is on the line.

The line determined by these points is shown in the Figure below.

(b) The slope is $m = 4.5$; it represents the rate of change of water level with respect to time. This means that the water level *increases* 4.5 ft per year. The $w$-intercept is 28 and occurs when $t = 0$, so it represents the water level when the dam was constructed.