Section 1.1 Real Numbers

Types of Real Numbers

1. **Natural numbers** ($\mathbb{N}$):

$$1,\ 2,\ 3,\ 4,\ 5,\ldots$$

2. **Integer numbers** ($\mathbb{Z}$):

$$0,\ \pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm5,\ldots$$

REMARK: Any natural number is an integer number, but not any integer number is a natural number.

3. **Rational numbers** ($\mathbb{Q}$):

$$r=\frac{m}{n},\quad \text{where } m\in\mathbb{Z}, n\in\mathbb{N}$$

EXAMPLES:

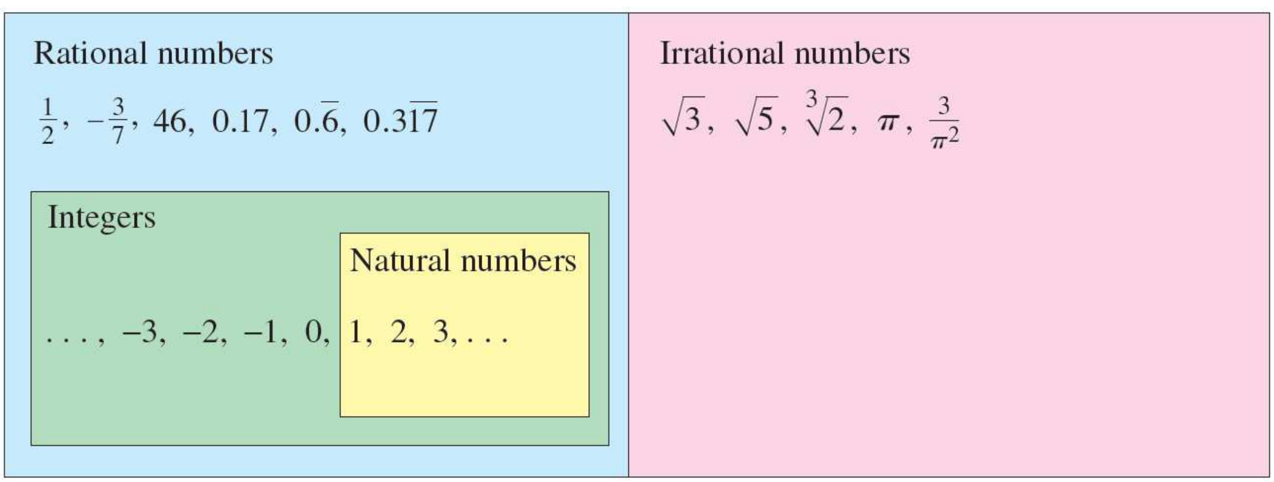
$$\frac{1}{2}\qquad \frac{7}{3}\qquad -\frac{11}{53}\qquad 2=\frac{2}{1}\qquad 0.2=\frac{2}{10}=\frac{1}{5}\qquad 0.222\ldots=0.\overline{2}=\frac{2}{9}\qquad 0.999\ldots=0.\overline{9}=1$$

REMARK: Any integer number is a rational number, but not any rational number is an integer.

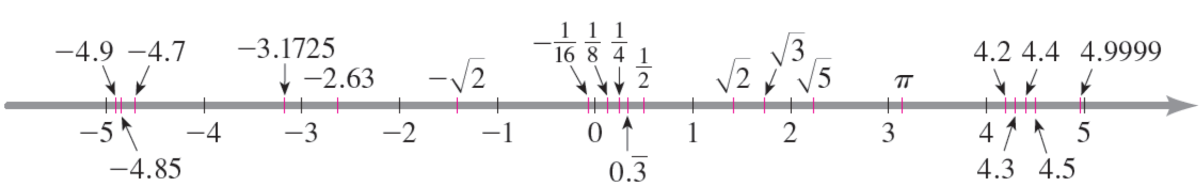
4. **Irrational Numbers**. These are numbers that cannot be expressed as a ratio of integers.

EXAMPLES:

$$\sqrt{2}\qquad \sqrt{3}\qquad \sqrt{2}+\sqrt{3}\qquad \frac{1+\sqrt{5}}{2}\qquad -\sqrt[3]{2}\qquad \pi\qquad \sqrt{\pi}\qquad 2^{\sqrt{2}}$$



The set of all real numbers is denoted by $\mathbb{R}$. The real numbers can be represented by points on a line which is called a **coordinate line**, or a **real number line**, or simply a **real line**:



Every real number has a decimal representation. If the number is rational, then its corresponding decimal is repeating. For example,

$$\frac{1}{2} = 0.500\ldots = 0.5\overline{0} = 0.4\overline{9}, \quad \frac{2}{3} = 0.66666\ldots = 0.\overline{6}$$

$$\frac{157}{495} = 0.3171717\ldots = 0.3\overline{17}, \quad \frac{9}{7} = 1.285714285714\ldots = 1.\overline{285714}$$

If the number is irrational, the decimal representation is nonrepeating:

$$\sqrt{2} = 1.414213562373095\ldots \qquad \pi = 3.141592653589793\ldots$$

Operations on Real Numbers

Real numbers can be combined using the familiar operations of addition, subtraction, multiplication, and division. When evaluating arithmetic expressions that contain several of these operations, we use the following conventions to determine the order in which the operations are performed:

1. Perform operations inside parentheses first, beginning with the innermost pair. In dividing two expressions, the numerator and denominator of the quotient are treated as if they are within parentheses.
2. Perform all multiplications and divisions, working from left to right.
3. Perform all additions and subtractions, working from left to right.

EXAMPLES:

1. $7 - 3 + 8 + 5 = 4 + 8 + 5 = 12 + 5 = 17$

2. $8 + 4 \cdot 2 = 8 + 8 = 16$

3. $9\left(\frac{24}{3} - \frac{49}{7}\right) + 5\left(\frac{56}{7} - \frac{32}{8}\right) = 9\,(8-7) + 5\,(8-4) = 9 \cdot 1 + 5 \cdot 4 = 9 + 20 = 29$

4. $3\left(\frac{8+10}{2 \cdot 3} + 4\right) - 2(5+9)$

4. We have

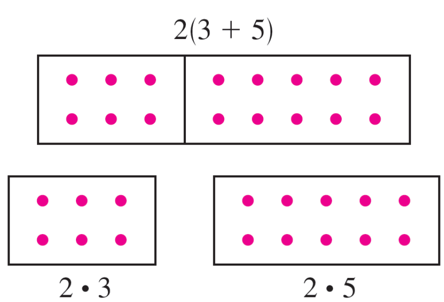
$$3\left(\frac{8+10}{2\cdot 3}+4\right)-2(5+9)=3\left(\frac{18}{6}+4\right)-2(5+9)$$
$$=3\,(3+4)-2(5+9)$$
$$=3\cdot 7-2\cdot 14$$
$$=21-28$$
$$=-7$$

Properties of Real Numbers

PROPERTIES OF REAL NUMBERS

Property	Example	Description
Commutative Properties		
$a+b=b+a$	$7+3=3+7$	When we add two numbers, order doesn't matter.
$ab=ba$	$3\cdot 5=5\cdot 3$	When we multiply two numbers, order doesn't matter.
Associative Properties		
$(a+b)+c=a+(b+c)$	$(2+4)+7=2+(4+7)$	When we add three numbers, it doesn't matter which two we add first.
$(ab)c=a(bc)$	$(3\cdot 7)\cdot 5=3\cdot(7\cdot 5)$	When we multiply three numbers, it doesn't matter which two we multiply first.
Distributive Property		
$a(b+c)=ab+ac$ $(b+c)a=ab+ac$	$2\cdot(3+5)=2\cdot 3+2\cdot 5$ $(3+5)\cdot 2=2\cdot 3+2\cdot 5$	When we multiply a number by a sum of two numbers, we get the same result as multiplying the number by each of the terms and then adding the results.

The Figure below explains why the Distributive Property works:



EXAMPLES:

1. $5(3+8)=5\cdot 3+5\cdot 8=15+40=55$

2. $(-b+3c)2a=(-b)\cdot 2a+3c\cdot 2a=-2ab+6ac$

3_1. $(a+b)(c+d)=a(c+d)+b(c+d)=ac+ad+bc+bd$

3_2. $(a+b)(c+d)=(a+b)c+(a+b)d=ac+bc+ad+bd$

Addition and Subtraction

The number 0 is special for addition; it is called the **additive identity** because

$$a + 0 = a$$

for any real number a. Every real number a has a **negative**, $-a$, that satisfies

$$a + (-a) = 0$$

Subtraction is the operation that undoes addition; to subtract a number from another, we simply add the negative of that number. By definition

$$a - b = a + (-b)$$

To combine real numbers involving negatives, we use the following properties.

Properties of Zero

Let a and b be real numbers, variables, or algebraic expressions.

1. $a + 0 = a$ and $a - 0 = a$
2. $a \cdot 0 = 0$
3. $\dfrac{0}{a} = 0, \qquad a \neq 0$
4. $\dfrac{a}{0}$ is undefined.
5. **Zero-Factor Property:** If $ab = 0$, then $a = 0$ or $b = 0$.

PROPERTIES OF NEGATIVES

Property	Example
1. $(-1)a = -a$	$(-1)5 = -5$
2. $-(-a) = a$	$-(-5) = 5$
3. $(-a)b = a(-b) = -(ab)$	$(-5)7 = 5(-7) = -(5 \cdot 7)$
4. $(-a)(-b) = ab$	$(-4)(-3) = 4 \cdot 3$
5. $-(a + b) = -a - b$	$-(3 + 5) = -3 - 5$
6. $-(a - b) = b - a$	$-(5 - 8) = 8 - 5$

EXAMPLES:

1_1. $-(8-15) \overset{(6)}{=} 15 - 8 = 7$

1_2. $-(8-15) = -(-7) \overset{(2)}{=} 7$

2. $-(-3+y-z) \overset{(5)}{=} -(-3) - y - (-z) \overset{(2)}{=} 3 - y + z$

Multiplication and Division

PROPERTIES OF FRACTIONS

Property	Example	Description
1. $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$	$\frac{2}{3}\cdot\frac{5}{7}=\frac{2\cdot5}{3\cdot7}=\frac{10}{21}$	When **multiplying fractions**, multiply numerators and denominators.
2. $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}$	$\frac{2}{3}\div\frac{5}{7}=\frac{2}{3}\cdot\frac{7}{5}=\frac{14}{15}$	When **dividing fractions**, invert the divisor and multiply.
3. $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$	$\frac{2}{5}+\frac{7}{5}=\frac{2+7}{5}=\frac{9}{5}$	When **adding fractions** with the **same denominator**, add the numerators.
4. $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$	$\frac{2}{5}+\frac{3}{7}=\frac{2\cdot7+3\cdot5}{35}=\frac{29}{35}$	When **adding fractions** with **different denominators**, find a common denominator. Then add the numerators.
5. $\frac{ac}{bc}=\frac{a}{b}$	$\frac{2\cdot5}{3\cdot5}=\frac{2}{3}$	**Cancel** numbers that are **common factors** in the numerator and denominator.
6. If $\frac{a}{b}=\frac{c}{d}$, then $ad=bc$	$\frac{2}{3}=\frac{6}{9}$, so $2\cdot9=3\cdot6$	**Cross multiply**.

EXAMPLES:

1. $\frac{\frac{1}{2}}{2}=\frac{\frac{1}{2}}{\frac{2}{1}}=\frac{1}{2}\div\frac{2}{1}=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{2\cdot2}=\frac{1}{4}$ or $\frac{\frac{1}{2}}{2}=\frac{\frac{1}{2}\cdot2}{2\cdot2}=\frac{1}{4}$

2. $\frac{2}{\frac{1}{2}}=\frac{\frac{2}{1}}{\frac{1}{2}}=\frac{2}{1}\div\frac{1}{2}=\frac{2}{1}\cdot\frac{2}{1}=2\cdot2=4$ or $\frac{2}{\frac{1}{2}}=\frac{2\cdot2}{\frac{1}{2}\cdot2}=\frac{4}{1}=4$

3. $\frac{\frac{2}{5}}{\frac{7}{10}}=\frac{2}{5}\cdot\frac{10}{7}=\frac{2\cdot10}{5\cdot7}=\frac{2\cdot2}{7}=\frac{4}{7}$

4_1. $\frac{\frac{2}{5}+\frac{1}{2}}{\frac{1}{10}+\frac{3}{15}}=\frac{\frac{2\cdot2+1\cdot5}{5\cdot2}}{\frac{1\cdot15+3\cdot10}{10\cdot15}}=\frac{\frac{9}{10}}{\frac{45}{150}}=\frac{9}{10}\cdot\frac{150}{45}=\frac{9\cdot150}{10\cdot45}=\frac{9\cdot15}{45}=\frac{9}{3}=3$

4_2. $\frac{\frac{2}{5}+\frac{1}{2}}{\frac{1}{10}+\frac{3}{15}}=\frac{\frac{2\cdot2+1\cdot5}{5\cdot2}}{\frac{1\cdot3}{10\cdot3}+\frac{3\cdot2}{15\cdot2}}=\frac{\frac{9}{10}}{\frac{1\cdot3+3\cdot2}{30}}=\frac{\frac{9}{10}}{\frac{9}{30}}=\frac{9}{10}\cdot\frac{30}{9}=\frac{9\cdot30}{10\cdot9}=3$

5. Evaluate $\frac{5}{36}+\frac{7}{120}$.

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Solution 1: We have

$$\frac{5}{36}+\frac{7}{120}=\frac{5\cdot 120+7\cdot 36}{36\cdot 120}=\frac{600+252}{4320}=\frac{852}{4320}=\frac{426}{2160}=\frac{213}{1080}=\frac{71}{360}$$

Solution 2_1: We have

$$\begin{aligned}\frac{5}{36}+\frac{7}{120}=\frac{5\cdot 120+7\cdot 36}{36\cdot 120}=\frac{5\cdot 2^3\cdot 3\cdot 5+7\cdot 2^2\cdot 3^2}{2^2\cdot 3^2\cdot 2^3\cdot 3\cdot 5}&=\frac{2^2\cdot 3(5\cdot 2\cdot 5+7\cdot 3)}{2^2\cdot 3^2\cdot 2^3\cdot 3\cdot 5}\\&=\frac{5\cdot 2\cdot 5+7\cdot 3}{3\cdot 2^3\cdot 3\cdot 5}\\&=\frac{50+21}{3\cdot 2^2\cdot 3\cdot 5\cdot 2}\\&=\frac{71}{36\cdot 10}\\&=\frac{71}{360}\end{aligned}$$

Solution 2_2: We have

$$36=2^2\cdot 3^2 \quad\text{and}\quad 120=2^3\cdot 3\cdot 5$$

We find the least common denominator (LCD) by forming the product of all the factors that occur in these factorizations, using the highest power of each factor. Thus, the LCD is $2^3{\cdot}3^2{\cdot}5=360$. So

$$\frac{5}{36}+\frac{7}{120}=\frac{5\cdot 10}{36\cdot 10}+\frac{7\cdot 3}{120\cdot 3}=\frac{50}{360}+\frac{21}{360}=\frac{71}{360}$$

6. We have

$$\frac{(7-6.35)\div 6.5+9.9}{\left(1.2\div 36+1.2\div 0.25-1\frac{5}{16}\right)\div\frac{169}{24}}=\frac{0.65\div 6.5+9.9}{\left(\frac{1}{30}+\frac{24}{5}-\frac{21}{16}\right)\cdot\frac{24}{169}}=\frac{0.1+9.9}{\frac{169}{48}\cdot\frac{24}{169}}=\frac{10}{\frac{1}{2}}=20$$

7. We have

$$\begin{aligned}\frac{\left(1\frac{1}{5}\div\left(\frac{17}{40}+0.6-0.005\right)\right)\cdot 1.7}{\frac{5}{6}+1\frac{1}{3}-1\frac{23}{30}}+\frac{4.75+7\frac{1}{2}}{33\div 4\frac{5}{7}}\div 0.25&=\frac{\frac{6}{5}\div\left(\frac{17}{40}+\frac{3}{5}-\frac{1}{200}\right)\cdot\frac{17}{10}}{\frac{5}{6}+\frac{4}{3}-\frac{53}{30}}+\frac{\frac{19}{4}+\frac{15}{2}}{33\cdot\frac{7}{33}}\cdot 4\\&=\frac{\frac{6}{5}\div\frac{51}{50}\cdot\frac{17}{10}}{\frac{2}{5}}+\frac{49}{4\cdot 7}\cdot 4=5+7=12\end{aligned}$$

Sets and Intervals

A **set** is a collection of objects, and these objects are called the **elements** of the set. If S is a set, the notation $a \in S$ means that a is an element of S, and $b \notin S$ means that b is not an element of S. For example, if $\mathbb{Q}$ represents the set of rational numbers, then $2 \in \mathbb{Q}$ but $\sqrt{2} \notin \mathbb{Q}$.

Some sets can be described by listing their elements within braces. For instance, the set A that consists of all positive integers less than 7 can be written as

$$A = \{1, 2, 3, 4, 5, 6\}$$

We could also write A in **set-builder notation** as

$$A = \{x \mid x \text{ is an integer and } 0 < x < 7\}$$

which is read "A is the set of all x such that x is an integer and $0 < x < 7$."

DEFINITION: If S and T are sets, then their **union** $S \cup T$ is the set that consists of all elements that are in S or T (or in both). The **intersection** of S and T is the set $S \cap T$ consisting of all elements that are in both S and T. The empty set, denoted by $\emptyset$, is the set that contains no element.

EXAMPLE: If $S = \{1, 2, 3, 4, 5\}$, $T = \{4, 5, 6, 7\}$, and $V = \{6, 7, 8\}$, find the sets $S \cup T$, $S \cap T$, and $S \cap V$.

Solution: We have

$$S \cup T = \{1, 2, 3, 4, 5, 6, 7\} \qquad S \cap T = \{4, 5\} \qquad S \cap V = \emptyset$$

EXAMPLE: If $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{2, 4, 6, 8\}$, and $C = \{7, 8, 9, 10\}$, find the sets $A \cup B$, $A \cap B$, $B \cup C$, $B \cap C$, $A \cup C$, $A \cap C$, $A \cup B \cup C$, and $A \cap B \cap C$.

EXAMPLE: If $A = \{1,2,3,4,5,6,7\}$, $B = \{2,4,6,8\}$, and $C = \{7,8,9,10\}$, find the sets $A \cup B$, $A \cap B$, $B \cup C$, $B \cap C$, $A \cup C$, $A \cap C$, $A \cup B \cup C$, and $A \cap B \cap C$.

Solution: We have

$A \cup B = \{1,2,3,4,5,6,7,8\}$ $\quad A \cap B = \{2,4,6\}$ $\quad B \cup C = \{2,4,6,7,8,9,10\}$ $\quad B \cap C = \{8\}$

$$A \cup C = \{1,2,3,4,5,6,7,8,9,10\} \qquad A \cap C = \{7\}$$

$$A \cup B \cup C = \{1,2,3,4,5,6,7,8,9,10\} \qquad A \cap B \cap C = \emptyset$$

Certain sets of real numbers, called **intervals**, occur frequently in calculus and correspond geometrically to line segments. If $a < b$, then the **open interval** from a to b consists of all numbers between a and b and is denoted by the symbol (a, b). The **closed interval** from a to b includes the endpoints and is denoted $[a, b]$. Using set-builder notation, we can write

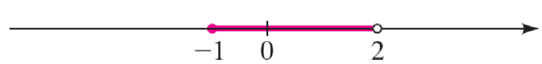
$$(a, b) = \{x \mid a < x < b\} \qquad [a, b] = \{x \mid a \le x \le b\}$$

The following table lists the possible types of intervals.

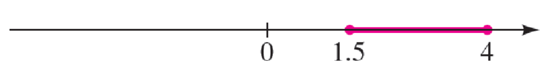
Notation	Set description	Graph
(a, b)	$\{x \mid a < x < b\}$	
$[a, b]$	$\{x \mid a \le x \le b\}$	
$[a, b)$	$\{x \mid a \le x < b\}$	
$(a, b]$	$\{x \mid a < x \le b\}$	
(a, ∞)	$\{x \mid a < x\}$	
$[a, \infty)$	$\{x \mid a \le x\}$	
$(-\infty, b)$	$\{x \mid x < b\}$	
$(-\infty, b]$	$\{x \mid x \le b\}$	
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

EXAMPLE: Express each interval in terms of inequalities, and then graph the interval:

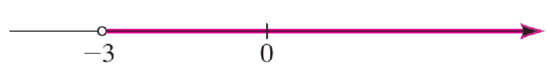
(a) $[-1, 2) = \{x \mid -1 \le x < 2\}$



(b) $[1.5, 4] = \{x \mid 1.5 \le x \le 4\}$

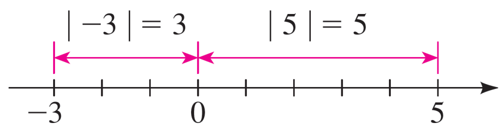


(c) $(-3, \infty) = \{x \mid -3 < x\}$



Absolute Value and Distance

The **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line.



Distance is always positive or zero, so we have $|a| \geq 0$ for every number a. Remembering that $-a$ is positive when a is negative, we have the following definition.

DEFINITION OF ABSOLUTE VALUE

If a is a real number, then the **absolute value** of a is

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

EXAMPLES:

(a) $|3| = 3$ (b) $|-3| = -(-3) = 3$ (c) $|0| = 0$ (d) $|3 - \pi| = -(3 - \pi) = \pi - 3$

PROPERTIES OF ABSOLUTE VALUE

Property	**Example**	**Description**
1. $\lvert a \rvert \geq 0$	$\lvert -3 \rvert = 3 \geq 0$	The absolute value of a number is always positive or zero.
2. $\lvert a \rvert = \lvert -a \rvert$	$\lvert 5 \rvert = \lvert -5 \rvert$	A number and its negative have the same absolute value.
3. $\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$	$\lvert -2 \cdot 5 \rvert = \lvert -2 \rvert \lvert 5 \rvert$	The absolute value of a product is the product of the absolute values.
4. $\left\lvert \frac{a}{b} \right\rvert = \frac{\lvert a \rvert}{\lvert b \rvert}$	$\left\lvert \frac{12}{-3} \right\rvert = \frac{\lvert 12 \rvert}{\lvert -3 \rvert}$	The absolute value of a quotient is the quotient of the absolute values.

DISTANCE BETWEEN POINTS ON THE REAL LINE

If a and b are real numbers, then the **distance** between the points a and b on the real line is

$$d(a, b) = |b - a|$$

Appendix

THEOREM: $\sqrt{2}$ is irrational.

Proof: Assume to the contrary that $\sqrt{2}$ is rational, that is

$$\sqrt{2} = \frac{p}{q}$$

where p and q are integers and $q \neq 0$. Without loss of generality we can assume that <u>p and q have no common divisor > 1</u>. Then

$$2 = \frac{p^2}{q^2} \quad \Longrightarrow \quad 2q^2 = p^2 \tag{1}$$

Since $2q^2$ is even, it follows that p^2 is even. Then <u>p is also even</u> (in fact, if p is odd, then p^2 is odd). This means that there exists $k \in \mathbb{Z}$ such that

$$p = 2k \tag{2}$$

Substituting (2) into (1), we get

$$2q^2 = (2k)^2 \quad \Longrightarrow \quad 2q^2 = 4k^2 \quad \Longrightarrow \quad q^2 = 2k^2$$

Since $2k^2$ is even, it follows that q^2 is even. Then <u>q is also even</u>. This is a contradiction. ■