Expectations and Decisions

Mathematical Expectation

EXAMPLE: Suppose we roll a fair, 6-sided die 100 times (keeping track of the results), and at the end, we add up all the results of each roll. What would be the likely value of this sum?

Solution: The sample space for a single roll can be described by the following:

Outcomes	1	2	3	4	5	6
Probabilities	1/6	1/6	1/6	1/6	1/6	1/6

Thus, the total sum should be about

$$s = 100 \times \frac{1}{6}(1) + 100 \times \frac{1}{6}(2) + 100 \times \frac{1}{6}(3) + 100 \times \frac{1}{6}(4) + 100 \times \frac{1}{6}(5) + 100 \times \frac{1}{6}(6)$$

$$= 100\left(\frac{1}{6}(1) + \frac{1}{6}(2) + \frac{1}{6}(3) + \frac{1}{6}(4) + \frac{1}{6}(5) + \frac{1}{6}(6)\right)$$

$$= 100 \times 3.5 = 350$$

EXAMPLE: Consider a lottery with a single JACKPOT prize of $500,000. If a ticket costs $3, and the probability of it being a winning ticket is approximately 1/1,000,000, then what is the expected winnings?

Solution: We have

Event	Winnings	Probability
win	$500,000-$3	1/1,000,000
lose	-$3	999,999/1,000,000

The expected winnings is

$$(\$499,997)\frac{1}{1,000,000} + (-\$3)\frac{999,999}{1,000,000} = -\$2.50$$

EXAMPLE: A contractor is bidding on a job that promises a profit of $200,000 with a probability of 7/10 and a loss, due to strikes, weather conditions, late arrival of building materials, and so on, of $40,000 with a probability of 3/10. Determine the contractor's expected profit.

Solution: We have

$$(\$200,000)\frac{7}{10} + (-\$40,000)\frac{3}{10} = \$128,000$$

Generalizing from this example leads to the following definition:

If the probabilities of obtaining the amounts $a_1, a_2, \ldots,$ or a_k are $p_1, p_2, \ldots,$ and p_k, where $p_1 + p_2 + \ldots + p_k = 1$, then the mathematical expectation is $E = a_1p_1 + a_2p_2 + \ldots + a_kp_k$.

EXAMPLE: What is our mathematical expectation if we win $25 when a die comes up 1 or 6 and lose $12.50 when it comes up 2,3,4, or 5?

EXAMPLE: What is our mathematical expectation if we win \$25 when a die comes up 1 or 6 and lose \$12.50 when it comes up 2,3,4, or 5?

Solution: We have

$$E = 25 \cdot \frac{1}{3} + (-12.5) \cdot \frac{2}{3} = 0$$

EXAMPLE: At a certain airport, the probabilities are $0.06, 0.21, 0.24, 0.18, 0.14, 0.10, 0.04, 0.02,$ and 0.01 that an airplane office will receive $0, 1, 2, 3, 4, 5, 6, 7,$ or 8 complaints per day about luggage handling. How many such complaints can this airplane office expect per day?

Solution: We have

$$E = 0(0.06) + 1(0.21) + 2(0.24) + 3(0.18) + 4(0.14) + 5(0.10) + 6(0.04) + 7(0.02) + 8(0.01) = 2.75$$

EXAMPLE: A school class of 120 students is driven somewhere in 3 buses. There are 36, 40, and 44 students on the bus 1, 2, and 3, respectively. Upon arrival, one of the 120 students is chosen randomly. What is the mathematical expectation of the number of students on the chosen students' bus?

Solution: We have

$$E = 36 \cdot \frac{36}{120} + 40 \cdot \frac{40}{120} + 44 \cdot \frac{44}{120} = 40.2667$$

Obviously, the average number of students per bus is $120/3 = 40$, showing the expected number of students on a randomly chosen students' bus is larger than the average. This is a general property. Basically, buses with many students are weighted heavily. This is easily seen if there were 118 students on one bus and 1 in each of the others. The expected number of students on a randomly chosen students' bus is then easily seen to be around 118.