

# Some Rules of Probability

## Sample Spaces and Events

We consider an experiment whose outcome is not predictable with certainty. However, we suppose that the set of all possible outcomes is known.

DEFINITION: The set of all possible outcomes of an experiment is the sample space of an experiment, and is denoted by  $S$ . Subsets of  $S$  are called events. Elements of  $S$ ,  $x \in S$ , are called outcomes.

EXAMPLES:

1. A coin is tossed twice and the outcome of each is recorded. Then,

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

The event that the second toss was a Head is the subset

$$E = \{(H, H), (T, H)\}$$

2. Consider 3 light-bulbs. Our experiment consists of finding out which light-bulb burns out first, and how long (in hours) it takes for this to happen.

$$S = \{(i, t) : i \in \{1, 2, 3\}, t \geq 0\}$$

$i$  tells you which one burns out, and  $t$  gives how long it lasted, in hours. The event that the 2nd bulb burns out first, and it lasts less than 3 hours is the set

$$E = \{(2, t) : t < 3\}$$

3. You roll a four sided die until a 4 comes up. The event you are interested in is getting a three with the first two rolls.

$$S = \{ a_1 a_2 \dots a_n \mid n \geq 1, a_n = 4, a_i \in \{1, 2, 3\} \text{ for } i \neq n \}$$

$$E = \{ 33a_3a_4 \dots a_n \mid n \geq 3, a_n = 4, a_i \in \{1, 2, 3\} \text{ for } i \neq n \}$$

DEFINITION: The union of  $E$  and  $F$ , denoted  $E \cup F$  or  $E + F$ , consists of those outcomes that are in either  $E$  or  $F$  (or both). (we write  $x \in E \cup F$  if  $x \in E$  or  $x \in F$ .)

EXAMPLE: If  $E = \{(H, H), (H, T)\}$  (Head on first toss) and  $F = \{(T, T), (H, T)\}$  (tail on the second toss). Then,

$$E \cup F = \{(H, H), (H, T), (T, T)\}$$

only missing  $(T, H)$ .

DEFINITION: The intersection of  $E$  and  $F$ , denoted  $E \cap F$  or  $EF$ , consists of those events that are in both  $E$  and  $F$ . (we write  $x \in EF$  if  $x \in E$  and  $x \in F$ .)

EXAMPLE: If  $E = \{(H, H), (H, T)\}$  (Head on the first toss) and  $F = \{(T, T), (H, T)\}$  (tail on the second toss). Then

$$EF = \{(H, T)\}$$

DEFINITION: The empty set  $\{\} = \emptyset$  is the set consisting of nothing.

DEFINITION: Two sets are exclusive if  $EF = \emptyset$ . The sets  $E_1, E_2, \dots$  are mutually exclusive if  $E_i E_j = \emptyset$  for all  $i \neq j$ .

EXAMPLE: Let  $S = \{(i, j) \mid i, j \in \{1, \dots, 6\}\}$  be the outcome from two rolls of a die. Let  $E$  be those such that  $i + j = 6$  (i.e.  $E = \{(1, 5), (2, 4), \dots\}$ ) and  $F$  be those such that  $i + j = 7$  (i.e.  $F = \{(1, 6), (2, 5), \dots\}$ ). Then  $EF = \emptyset$  and they are exclusive.

DEFINITION: The complement of  $E$ , denoted  $E^c$  or  $E'$ , consists of those outcomes that are not in  $E$ . Note that  $S^c = \emptyset$  and that  $(E^c)^c = E$ .

DEFINITION:  $E$  is a subset of  $F$  if  $x \in E$  implies  $x \in F$ . Notation:  $E \subset F$ .

DEFINITION:  $E$  and  $F$  are equal, denoted  $E = F$ , if  $E \subset F$  and  $F \subset E$ .

## The Postulates of Probability

Let  $S$  be a sample space for some experiment. For each event  $E \subset S$ , we assume that a number  $P(E)$  is defined and satisfies the following three postulates:

Postulate 1:  $0 \leq P(E) \leq 1$ .

Postulate 2:  $P(S) = 1$ .

Postulate 3: If  $\{E_1, E_2\}$  are exclusive, then  $P(E_1 \cup E_2) = P(E_1) + P(E_2)$ .

Generalization of Postulate 3: If  $\{E_i\}$  are mutually exclusive ( $E_i \cap E_j = \emptyset$  if  $i \neq j$ ), then  $P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$ . We call  $P(A)$  the probability of  $E$ .

EXAMPLES:

1. Consider flipping a fair coin and recording the outcome. Then  $S = \{H, T\}$  and

$$P(\{H\}) = 1/2$$

$$P(\{T\}) = 1/2$$

$$P(\{H, T\}) = 1$$

Note that

$$P(\{H\} \cup \{T\}) = 1 = 1/2 + 1/2 = P(\{H\}) + P(\{T\})$$

This is not the only probability possible for this experiment/sample space. Could have an unfair coin.

$$P(\{H\}) = p, \quad P(\{T\}) = 1 - p, \quad P(\{H, T\}) = 1$$

2. Rolling a die. Assuming that the die is fair,  $P(\{1\}) = P(\{2\}) = \dots = P(\{6\}) = 1/6$ . Probability of rolling an even number is

$$P(\{2, 4, 6\}) = P(\{2\}) + P(\{4\}) + P(\{6\}) = 1/6 + 1/6 + 1/6 = 1/2$$

## Addition Rules

THEOREM:  $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

EXAMPLE: In a sample survey conducted in a suburban community, the probabilities are 0.92, 0.53 and 0.48 that a family selected at random will own a family sedan, a sports utility vehicle, or both. What is the probability that such a family will own a family sedan, a sports utility vehicle, or both?

Solution: Let  $S$  be the event that a family will own a family sedan and let  $V$  be the event that a family will own a sports utility vehicle. Then

$$P(S \cup V) = P(S) + P(V) - P(S \cap V) = 0.92 + 0.53 - 0.48 = 0.97$$

EXAMPLE: A retail establishment accepts either the American Express or the VISA credit card. A total of 24 percent of its customers carry an American Express card, 61 percent carry a VISA card, and 11 percent carry both. What percentage of its customers carry a credit card that the establishment will accept?

Solution: The establishment will accept the credit card if it is American Express or VISA. Let  $A$  be the event that a customer has American Express and let  $V$  be the event that a customer has a VISA card. Then

$$P(A \cup V) = P(A) + P(V) - P(A \cap V) = 0.24 + 0.61 - 0.11 = 0.74$$

EXAMPLE: A number is chosen at random from  $S = \{1, \dots, 520\}$ . What is the probability that it is divisible by 5 or 7?

Solution: Assume each number has a probability of  $1/520$  of being chosen. Let  $A$  be the event that it is divisible by 5. Let  $B$  be the event that it is divisible by 7. We have  $520/5 = 104$ , so  $P(A) = 104/520$ . Similarly,  $520/7 = 74 + 2/7$ , so  $P(B) = 74/520$ . Want  $P(A \cup B)$ . So, by the theorem, we need  $P(A \cap B)$ .  $x \in A \cap B$  if and only if divisible by 35.  $520/35 = 14 + 30/35$ . So  $P(A \cap B) = 14/520$ . Thus,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 104/520 + 74/520 - 14/520 = 164/520$$

THEOREM:  $P(E^c) = 1 - P(E)$

THEOREM:  $(E \cup F)^c = E^c \cap F^c$

EXAMPLE: You believe the following: with probability 0.5 it will snow today. With probability 0.3 it will snow tomorrow. With probability 0.2 it will snow both days. What is the probability that it will not snow either day?

Solution: Let  $E, F$  be the events that it snows today/tomorrow, resp. Then,

$$P(E) = 0.5, \quad P(F) = 0.3, \quad \text{and} \quad P(E \cap F) = 0.2$$

We need  $P(E^c \cap F^c)$ . Not snowing,  $E^c \cap F^c$ , is the complement of snowing at least once,  $E \cup F$ , and so,

$$\begin{aligned} P(E^c \cap F^c) &= P((E \cup F)^c) = 1 - P(E \cup F) = 1 - (P(E) + P(F) - P(E \cap F)) \\ &= 1 - (0.5 + 0.3 - 0.2) = 1 - 0.6 = 0.4 \end{aligned}$$

# Conditional Probability

EXAMPLES:

1. Consider rolling a die. We know that  $S = \{1, 2, 3, 4, 5, 6\}$ . Let  $E$  be the event that the outcome is in  $\{2, 3, 6\}$ . We know that  $P(E) = 1/2$ . Now suppose I tell you that the roll was an even number. Given this knowledge, what is the probability that  $E$  occurred?

Solution: Let  $F = \{2, 4, 6\}$ . Since we know that each of 2, 4, 6 is equally likely, the probability of  $E$  now appears to be

$$\text{Cond. Prob. of 2} + \text{Cond. Prob. of 3} + \text{Cond. Prob. of 6} = \frac{1}{3} + 0 + \frac{1}{3} = \frac{2}{3}$$

2. Now consider rolling an unfair die. We know that  $S = \{1, 2, 3, 4, 5, 6\}$ , but assume that  $P(\{i\}) = 0.1$  for  $i \in \{1, 2, 3, 4, 5\}$  and  $P(\{6\}) = 0.5$ . Let  $E$  be the event that the outcome is in  $\{2, 3, 6\}$ . We know that

$$P(E) = P(\{2\}) + P(\{3\}) + P(\{6\}) = 0.7$$

Now suppose I tell you that the roll was an even number. Given this knowledge, what is the probability that  $E$  occurred?

Solution: Let  $F = \{2, 4, 6\}$ . Clearly, the outcome  $\{3\}$  did not occur, so that has zero probability now. What about  $\{2\}$  and  $\{6\}$ ? We know that  $P(F) = 0.7$  and that  $P(\{2\}) = 0.1$ ,  $P(\{4\}) = 0.1$ ,  $P(\{6\}) = 0.5$ . Thus, it seems like

$$\text{Cond. Prob. of 2} + \text{Cond. Prob. of 3} + \text{Cond. Prob. of 6} = \frac{0.1}{0.7} + 0 + \frac{0.5}{0.7} = \frac{6}{7}$$

How did we intuitively compute that? For  $E$  to occur given that  $F$  already has, the outcome *must* be in  $EF$  (so you intuitively ignored the outcome “3” in  $E$ ). Because we know that  $F \subset S$  has occurred, we make  $F$  our new, or *reduced* sample space. Next, the probability of  $EF$  occurring given that  $F$  has should be the probability of  $EF$  relative to the probability of  $F$ : i.e. the probability of  $\{2, 6\}$  relative to  $\{2, 4, 6\}$  (consider Venn diagrams). That is,

DEFINITION: Let  $E, F \subset S$  with  $P(F) > 0$ , then

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

In the first example above,  $E \cap F = \{2, 6\}$  and so

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(\{2, 6\})}{P(\{2, 4, 6\})} = \frac{2/6}{3/6} = \frac{2}{3}$$

The above is taken to be the definition of conditional probability in all cases.

EXAMPLES:

1. The probability that a person lives to 80 is 0.64. The probability that a person lives until 90 is 0.51. What is the probability that a person will live to 90, given that that person is 80?

1. The probability that a person lives to 80 is 0.64. The probability that a person lives until 90 is 0.51. What is the probability that a person will live to 90, given that that person is 80?

Solution: Let  $A$  be the event of living until 90 and  $B$  be the event of living until 80. Note that  $A = A \cap B$ . Then,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{0.51}{0.64} \approx 0.797$$

2. At a certain elementary school, the probability that a student selected at random will come from a one-parent home is 0.36 and the probability that he or she will come from a one-parent home and also be a low achiever (gets mostly  $D$ 's and  $F$ 's) is 0.27. What is the probability that such a randomly selected student will be a low achiever given that he or she comes from a one-parent home?

Solution: If we let  $L$  denote a low achiever and  $O$  a student from a one-parent home, we have  $P(O) = 0.36$  and  $P(O \cap L) = 0.27$ , and we get

$$P(L|O) = \frac{P(O \cap L)}{P(O)} = \frac{0.27}{0.36} = 0.75$$

3. A coin is flipped twice. Assuming all points of  $S = \{(H, H), (T, H), (H, T), (T, T)\}$  are equally likely, what is the conditional probability that both flips are heads, given that

(a) the first flip is a head?

(b) at least one flip lands on heads?

Solution: Let  $E$  be the event that both flips are heads. Let  $A$  be the event that the first flip is a head. Then

$$P(E|A) = \frac{P(EA)}{P(A)} = \frac{P(\{(H, H)\})}{P(\{(H, T), (H, H)\})} = \frac{1/4}{2/4} = \frac{1}{2}$$

Next, let  $B$  be the event that at least one flip is a head. Then

$$P(E|B) = \frac{P(EB)}{P(B)} = \frac{P(\{(H, H)\})}{P(\{(H, T), (H, H), (T, H)\})} = \frac{1/4}{3/4} = \frac{1}{3}$$

This seems counter-intuitive as “the other could be heads or tails.” So it seems that answer should be  $1/2$ . However, the information “at least one coin landed on heads” only tells us that  $(T, T)$  didn't happen. The remaining three options are equally likely.

4. The probability that Henry will like a new movie is 0.70 and the probability that Jean, his girlfriend, will like it is 0.60. If the probability is 0.28 that he will like it and she will dislike it, what is the probability that he will like it given that she is not going to like it?

Solution: If  $H$  and  $J$  are the events that Henry will like the new movie and that Jean will like it, we have

$$P(J') = 1 - 0.60 = 0.40 \quad \text{and} \quad P(H \cap J') = 0.28$$

therefore

$$P(H|J') = \frac{P(H \cap J')}{P(J')} = \frac{0.28}{0.40} = 0.70$$

What is special and interesting about the above result is that  $P(H)$  and  $P(H|J')$  both equal 0.70. Moreover, one can show that  $P(H|J)$  is also equal to 0.70. Thus, the probability of event  $H$  is the same regardless of whether or not event  $J$  has occurred, occurs, or will occur, and we say that event  $H$  is **independent** of event  $J$ . In general, it can easily be verified that when one event is independent of another, the second event is also independent of the first, and we say that the two events are **independent**. Two events  $A$  and  $B$  are independent if and only if

$$\boxed{P(A \cap B) = P(A)P(B)}$$

When two events are not independent, we say that they are **dependent**.

EXAMPLE: The chess clubs of two schools consist of, respectively, 8 and 9 players. Four members from each club are randomly chosen to participate in a contest between two schools. The chosen players from one team are then randomly paired with those from the other team, and each pairing plays a game of chess. Suppose that Rebecca and her sister Elise are on the chess clubs at different schools. What is the probability that either Rebecca or Elise will be chosen to represent her school?

Solution: Let  $R$  be the event that Rebecca is chosen to represent her school and let  $E$  be the event that Elise is chosen to represent her school. We are interested in calculating  $P(R \cup E)$ . We have

$$P(R \cup E) = P(R) + P(E) - P(R \cap E) = \frac{4}{8} + \frac{4}{9} - \frac{4}{8} \cdot \frac{4}{9} = \frac{13}{18}$$

## Multiplication Rule

We have

$$P(E|F) = \frac{P(EF)}{P(F)}$$

Therefore,

$$\boxed{P(EF) = P(F)P(E|F)}$$

Thus, the probability of  $E$  and  $F$  occurring is equal to the probability of  $F$  occurring times the probability that  $E$  occurs given that  $F$  has.

EXAMPLE: Suppose an urn contains 8 red balls and 4 white balls. We draw two balls without replacement. If we assume that at each draw each ball in the urn is equally likely to be chosen, what is the probability that both balls drawn are red?

Solution: Let  $R_1$  and  $R_2$  be the events that the first and second ball are red, respectively. Then we want  $P(R_1R_2)$ . Note that we easily have that  $P(R_1) = 8/12$ . Also,  $P(R_2|R_1) = 7/11$  because the *reduced* sample space  $R_1$  consists of 7 red balls and 4 white balls, and we are asking for the probability of choosing a red ball. Therefore,

$$P(R_1R_2) = P(R_1)P(R_2|R_1) = \frac{8}{12} \cdot \frac{7}{11} = \frac{56}{132} = \frac{14}{33}$$

Note, could also just do straightaway as  $\binom{8}{2} / \binom{12}{2}$ .