Possibilities and Probabilities

Counting

The Basic Principle of Counting: Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes and if, for each outcome of experiment 1, there are n possible outcomes of experiment 2, then together there are $\boxed{mn}$ possible outcomes of the two experiments.

Proof: For $1 \leq i \leq m$ and $1 \leq j \leq n$ we say the outcome of the two experiments was (i, j) if the outcome of experiment 1 was i and experiment 2 was j. Now, simply write down all possible outcomes in the following way:

$$\begin{array}{llll} (1,1), & (1,2), & \ldots, & (1,n) \\ (2,1), & (2,2), & \ldots, & (2,n) \\ \ldots & & & \\ (m,1), & (m,2), & \ldots, & (m,n) \end{array}$$

Multiplying the number of rows (m) by the number of columns (n) shows there are a total number of mn possible outcomes.

EXAMPLES:

1. At a high school there are 12 teachers, with each teaching 4 courses. One teacher is to be given an award for best teaching *for a given course.* How many different choices of teacher/course are possible?

Solution: We regard the choice of teacher as the outcome of the first experiment. The outcome of the second experiment is one of the four courses that teacher is teaching. From the basic principle of counting we see there are $12 \times 4 = 48$ possible choices.

2. A small community consists of 10 women, each of whom has 3 children. If one woman and one of her children are to be chosen as mother and child of the year, how many different choices are possible?

Solution: By regarding the choice of the woman as the outcome of the first experiment and the subsequent choice of one of her children as the outcome of the second experiment, we see from the basic principle that there are $10 \times 3 = 30$ possible choices.

The Generalized Basic Principle of Counting: If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes; and if, for each of these n_1 possible outcomes, there are n_2 possible outcomes of the second experiment; and if, for each of the possible outcomes of the first two experiments, there are n_3 possible outcomes of the third experiment; and if..., then there are a total of $\boxed{n_1 n_2 \ldots n_r}$ possible outcomes of the r experiments.

PROBLEMS:

1. Consider rolling 7 dice. How many possible outcomes are there?

2. There are 4 math, 3 chemistry, and 2 Spanish books on a shelf and you need one of each for the coming semester. How many ways can you choose one of each?

3. How many functions defined on n points are possible if each functional value is either 0 or 1?

4. Suppose a state's license plate consists of 3 numbers followed by 3 letters. However, no two letters or numbers can be the same (i.e. no repetition is allowed). How many different license plates can be made?

PROBLEMS:

1. Consider rolling 7 dice. How many possible outcomes are there?

Solution: For $i \leq 7$, let E_i be the possible outcomes of the ith die ($= \{1, 2, \ldots, 6\}$). Therefore, $n_i = 6$ for each i. By the above theorem, the total number of possibilities is

$$6 \times 6 \times \ldots \times 6 = 6^7$$

2. There are 4 math, 3 chemistry, and 2 Spanish books on a shelf and you need one of each for the coming semester. How many ways can you choose one of each?

Solution: View the first experiment as choosing one of the math books, etc. There are

$$4 \times 3 \times 2 = 24$$

possibilities.

3. How many functions defined on n points are possible if each functional value is either 0 or 1?

Solution: Let the points be $1, 2, \ldots, n$. Since $f(i)$ must be either 0 or 1 for each $i = 1, 2, \ldots, n$, it follows that there are 2^n possible functions.

4. Suppose a state's license plate consists of 3 numbers followed by 3 letters. However, no two letters or numbers can be the same (i.e. no repetition is allowed). How many different license plates can be made?

Solution: The first number can be any of $\{0, 1, \ldots, 9\}$. So $n_1 = 10$. However, the second number can not be the first and so $n_2 = 9$. Similarly $n_3 = 8, n_4 = 26, n_5 = 25, n_6 = 24$. Thus, there are a total number of

$$10 \times 9 \times 8 \times 26 \times 25 \times 24 = 11,232,000$$

Permutations

DEFINITION: A *permutation* of a set X is a rearrangement of its elements.

EXAMPLES:

1. Let $X = \{1, 2\}$. Then there are 2 permutations:

$$12, \ 21$$

2. Let $X = \{1, 2, 3\}$. Then there are 6 permutations:

$$123, \ 132, \ 213, \ 231, \ 312, \ 321$$

3. Let $X = \{1, 2, 3, 4\}$. Then there are 24 permutations:

$$\begin{array}{cccccc} 1234, & 1243, & 1324, & 1342, & 1423, & 1432 \\ 2134, & 2143, & 2314, & 2341, & 2413, & 2431 \\ 3214, & 3241, & 3124, & 3142, & 3421, & 3412 \\ 4231, & 4213, & 4321, & 4312, & 4123, & 4132 \end{array}$$

REMARK: One can show that there are exactly $n!$ permutations of the n-element set X. In fact, we can use the generalized basic principle of counting. There are n possible outcomes of experiment 1, $n - 1$ for the second, etc. Thus, there are

$$\boxed{n! = n(n-1)(n-2)\ldots 2 \cdot 1}$$

permutations of n objects.

EXAMPLES:

1. In a class with 8 students, how many ways can the students be lined up?

Solution: There are $8! = 40,320$ ways to line up the students.

2. A class consists of 15 students, 6 boys and 9 girls. Suppose there are also 3 instructors. Suppose that we want everyone to line up at random.

(a) How many ways can they line up?

(b) How many ways can they line up so that all the boys will be together, all the girls will be together, and all the instructors will be together?

2. A class consists of 15 students, 6 boys and 9 girls. Suppose there are also 3 instructors. Suppose that we want everyone to line up at random.

(a) How many ways can they line up?

(b) How many ways can they line up so that all the boys will be together, all the girls will be together, and all the instructors will be together?

Solution:

(a) There are a total of $18! \approx 6.4 \times 10^{15}$ ways for everyone to line up.

(b) There are $6!, 9!$, and $3!$ permutations of the boys, girls, and instructors respectively. Therefore, by the basic counting principle, there are $6!9!3!$ ways for them to line up with all boys, then girls, then instructors. But, that was one permutation of boys, girls, and instructors. There are $3!$ permutations of those sets (i.e. Boys/Girls/Inst., Girls/Boys/Inst., etc.). Each of those orderings also has $6!9!3!$ possible arrangements. Thus, the total number is $3!$ times $6!9!3!$, or $3!6!9!3! \approx 9.4 \times 10^9$.

3. How many different letter arrangements can be formed from the letters $PEPPER$?

Solution: First note there are a total of $6!$ different letter permutations. Pick one: $EPEPPR$. Note that we could permute the E's and not change the word, and we could permute the P's and not change the word. There are $2!$ ways to permute the E's and $3!$ ways to permute the P's. Multiplying (using the counting principle) shows that there are $3!2!$ permutations of the letters in $EPEPPR$ that do not change the word. Thus, this arrangement has been counted $3!2!$ times (in getting to the $6!$ number of total permutations), and so has every other word. Thus, there are

$$\frac{6!}{3!2!} = 60$$

unique letter arrangements.

4. Consider various ways of ordering the letters in the word $MISSISSIPPI$:

$$IIMSSPISSIP,\ ISSSPMIIPIS,\ PIMISSSSIIP, \quad \text{and so on.}$$

How many distinguishable orderings are there?

4. Consider various ways of ordering the letters in the word $MISSISSIPPI$:

$$IIMSSPISSIP, \ ISSSPMIIPIS, \ PIMISSSSIIP, \quad \text{and so on.}$$

How many distinguishable orderings are there?

Solution 1: We first note that there are 11! permutations of the letters $MI_1S_1S_2I_2S_3S_4I_3P_1P_2I_4$ when the 4 I's, 4 S's and the 2 P's are distinguished from each other. However, consider any one of these permutations — for instance, $I_3P_2P_1I_4MI_2S_2S_1I_1S_3S_4$. If we now permute the I's among themselves, S's among themselves and P's among themselves, then the resultant arrangement would still be of the form $IPPIMISSISS$. That is, all 4!4!2! permutations are of the form $IPPIMISSISS$. Hence, there are

$$\frac{11!}{4!4!2!} = 34,650$$

possible letter arrangements of the letters $MISSISSIPPI$.

Solution 2: Since there are 11 positions in all, there are

$$\binom{11}{4} \text{ subsets of 4 positions for the } S\text{'s.}$$

Once the four S's are in place, there are

$$\binom{7}{4} \text{ subsets of 4 positions for the } I\text{'s.}$$

After the I's are in place, there are

$$\binom{3}{2} \text{ subsets of 2 positions for the } P\text{'s.}$$

That leaves just one position for the M. Hence, by the generalized basic principle of counting we have:

$$\begin{bmatrix} \text{number of ways to} \\ \text{position all the letters} \end{bmatrix} = \binom{11}{4} \cdot \binom{7}{4} \cdot \binom{3}{2} = 34,650$$

THEOREM: In general, there are

$$\boxed{\frac{n!}{n_1!n_2!\dots n_r!}}$$

different permutations of n objects, of which n_1 are alike, n_2 are alike, $\dots$, n_r are alike.

EXAMPLE: How many different 10-letter codes can be made using three a's, four b's, and three c's?

Solution: By the Theorem above we have

$$\frac{10!}{3!4!3!} = 4,200$$

Combinations

DEFINITION: We define $\binom{n}{r}$, for $r \leq n$, by

$$\boxed{\binom{n}{r} = \frac{n!}{r!(n-r)!}}$$

and say that $\binom{n}{r}$ represents the number of possible combinations of n objects taken r at a time.

EXAMPLES: We have

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4!}{2!\cdot 2!} = \frac{1\cdot 2\cdot 3\cdot 4}{(1\cdot 2)\cdot(1\cdot 2)} = \frac{3\cdot 4}{1\cdot 2} = 6$$

$$\binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4!}{3!\cdot 1!} = \frac{1\cdot 2\cdot 3\cdot 4}{(1\cdot 2\cdot 3)\cdot 1} = \frac{4}{1} = 4$$

$$\binom{4}{4} = \frac{4!}{4!(4-4)!} = \frac{4!}{4!\cdot 0!} = \frac{1\cdot 2\cdot 3\cdot 4}{(1\cdot 2\cdot 3\cdot 4)\cdot 1} = \frac{1}{1} = 1$$

$$\binom{8}{5} = \frac{8!}{5!(8-5)!} = \frac{8!}{5!\cdot 3!} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}{(1\cdot 2\cdot 3\cdot 4\cdot 5)\cdot(1\cdot 2\cdot 3)} = \frac{6\cdot 7\cdot 8}{1\cdot 2\cdot 3} = \frac{7\cdot 8}{1} = 56$$

EXAMPLES:

1. Suppose 5 members of a group of 12 are to be chosen to work as a team on a special project. How many distinct 5-person teams can be formed?

Solution: The number of distinct 5-person teams is the same as the number of subsets of size 5 (or 5-combinations) that can be chosen from the set of 12. This number is

$$\binom{12}{5} = \frac{12!}{5!(12-5)!} = \frac{12!}{5!\cdot 7!} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11\cdot 12}{(1\cdot 2\cdot 3\cdot 4\cdot 5)\cdot(1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7)}$$

$$= \frac{8\cdot 9\cdot 10\cdot 11\cdot 12}{1\cdot 2\cdot 3\cdot 4\cdot 5} = \frac{8\cdot 9\cdot 10\cdot 11\cdot 12}{(2\cdot 5)\cdot(3\cdot 4)} = \frac{8\cdot 9\cdot 10\cdot 11\cdot 12}{10\cdot 12} = 8\cdot 9\cdot 11 = 792$$

2. Suppose two members of the group of 12 refuse to work together on a team. How many distinct 5-person teams can be formed?

2. Suppose two members of the group of 12 refuse to work together on a team. How many distinct 5-person teams can be formed?

Solution 1: Call the two members of the group that refuse to work together A and B. We have

$$\begin{aligned}
&\begin{bmatrix} \text{number of 5-person teams} \\ \text{that do not contain} \\ \text{both } A \text{ and } B \end{bmatrix} \\
&= \begin{bmatrix} \text{number of 5-person} \\ \text{teams that contain} \\ A \text{ but not } B \end{bmatrix} + \begin{bmatrix} \text{number of 5-person} \\ \text{teams that contain} \\ B \text{ but not } A \end{bmatrix} + \begin{bmatrix} \text{number of 5-person} \\ \text{teams that contain} \\ \text{neither } A \text{ nor } B \end{bmatrix} \\
&= \binom{10}{4} + \binom{10}{4} + \binom{10}{5} = 210 + 210 + 252 = 672
\end{aligned}$$

since

$$\binom{10}{4} = \frac{10!}{4!(10-4)!} = \frac{10!}{4!\cdot 6!} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10}{(1\cdot 2\cdot 3\cdot 4)\cdot(1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6)} = \frac{7\cdot 8\cdot 9\cdot 10}{1\cdot 2\cdot 3\cdot 4} = \frac{7\cdot 9\cdot 10}{1\cdot 3}$$

$$= 7\cdot 3\cdot 10 = 210$$

and

$$\binom{10}{5} = \frac{10!}{5!(10-5)!} = \frac{10!}{5!\cdot 5!} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10}{(1\cdot 2\cdot 3\cdot 4\cdot 5)\cdot(1\cdot 2\cdot 3\cdot 4\cdot 5)} = \frac{6\cdot 7\cdot 8\cdot 9\cdot 10}{1\cdot 2\cdot 3\cdot 4\cdot 5}$$

$$= \frac{7\cdot 8\cdot 9\cdot 10}{4\cdot 5} = 7\cdot 2\cdot 9\cdot 2 = 252$$

Solution 2: We have

$$\begin{aligned}
\begin{bmatrix} \text{number of 5-person teams} \\ \text{that do not contain} \\ \text{both } A \text{ and } B \end{bmatrix} &= \begin{bmatrix} \text{total number of} \\ \text{5-person teams} \end{bmatrix} - \begin{bmatrix} \text{number of 5-person teams} \\ \text{that contain} \\ \text{both } A \text{ and } B \end{bmatrix} \\
&= \binom{12}{5} - \binom{10}{3} = 792 - 120 = 672
\end{aligned}$$

since $\binom{12}{5} = 792$ by the Example above and

$$\binom{10}{3} = \frac{10!}{3!(10-3)!} = \frac{10!}{3!\cdot 7!} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10}{(1\cdot 2\cdot 3)\cdot(1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7)} = \frac{8\cdot 9\cdot 10}{1\cdot 2\cdot 3}$$

$$= 4\cdot 3\cdot 10 = 120$$

3. Suppose the group of 12 consists of 5 men and 7 women. How many 5-person teams can be chosen that consist of 3 men and 2 women?

3. Suppose the group of 12 consists of 5 men and 7 women. How many 5-person teams can be chosen that consist of 3 men and 2 women?

Solution: Note, that there are $\binom{5}{3}$ ways to choose the three men out of the five and $\binom{7}{2}$ ways to choose the two women out of the seven. Therefore, by the generalized basic principle of counting we have

$$\begin{bmatrix} \text{number of 5-person teams that} \\ \text{contain 3 men and 2 women} \end{bmatrix} = \binom{5}{3} \cdot \binom{7}{2} = 10 \cdot 21 = 210$$

since

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3! \cdot 2!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(1 \cdot 2 \cdot 3) \cdot (1 \cdot 2)} = \frac{4 \cdot 5}{1 \cdot 2} = 2 \cdot 5 = 10$$

and

$$\binom{7}{2} = \frac{7!}{2!(7-2)!} = \frac{7!}{2! \cdot 5!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}{(1 \cdot 2) \cdot (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5)} = \frac{6 \cdot 7}{1 \cdot 2} = 3 \cdot 7 = 21$$

Properties of Binomial Coefficients

1. $\binom{n}{0} = \binom{n}{n} = 1$

Proof: We have

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \cdot n!} = 1 \qquad \binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n! \cdot 0!} = \frac{n!}{n! \cdot 1} = 1$$

2. $\binom{n}{1} = \binom{n}{n-1} = n$

Proof: We have

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{(n-1)! \cdot n}{1! \cdot (n-1)!} = n$$

$$\binom{n}{n-1} = \frac{n!}{(n-1)![n-(n-1)]!} = \frac{n!}{(n-1)! \cdot 1!} = \frac{(n-1)! \cdot n}{(n-1)! \cdot 1!} = n$$

3. $\binom{n}{k} = \binom{n}{n-k}$

Proof: We have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \frac{n!}{(n-k)![n-(n-k)]!} = \binom{n}{n-k}$$

4. $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$

Proof: We have

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\
&= \frac{n!k}{(k-1)!k(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k)!(n-k+1)} \\
&= \frac{n!k}{k!(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k+1)!} \\
&= \frac{n!k + n!(n-k+1)}{k!(n-k+1)!} = \frac{n!(k+n-k+1)}{k!(n-k+1)!} \\
&= \frac{n!(n+1)}{k!(n-k+1)!} = \frac{(n+1)!}{k!(n-k+1)!} = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}
\end{aligned}$$

Probability

The most common way of measuring uncertainties connected with events is to assign them **probabilities** or to specify the **odds** at which it would be fair to bet that the events will occur. Historically, the oldest method of measuring uncertainties is the **classical probability concept**.

> **If there are n equally likely possibilities, of which one must occur and s are regarded as favorable, or as a "success", then the probability of a "success" is $\frac{s}{n}$.**

EXAMPLE: What is the probability of drawing an ace from an ordinary deck of 52 playing cards?

Solution: Since there are $n = 4$ aces among the $N = 52$ cards, the probability of drawing an ace is $\frac{4}{52} = \frac{1}{13}$.

Although equally likely possibilities are found mostly in games of chance, the classical probability concept applies also in a great variety of situations where gambling devices are used to make **random selections** — when office space is assigned to teaching assistants by lot, when some of the families in a township are chosen in such a way that each one has the same chance of being included in a sample study, when machine parts are chosen for inspection so that each part produced has the same chance of being selected, etc.

EXAMPLE: A committee of 6 is to be chosen from 15 people. There are 5 Americans, 3 Canadians, and 7 Europeans in the group. Assuming the committee is randomly chosen, what is the probability that the committee will consist of 2 Americans, 1 Canadian, and 3 Europeans?

Solution: There are a total of $\binom{15}{6}$ possible committees. There are $\binom{5}{2}$, $\binom{3}{1}$, and $\binom{7}{3}$ ways to choose 2 Americans, 1 Canadian, and 3 Europeans, respectively. Therefore, by the Generalized Basic Principle of Counting, the probability is

$$\frac{\binom{5}{2}\binom{3}{1}\binom{7}{3}}{\binom{15}{6}} = \frac{30}{143} \approx 0.2097$$

EXAMPLE: In the Senate, there are 53 liberals and 47 conservatives. A committee is composed randomly of 9 senators. We know that the committee does not contain more than 3 conservative senators. What is the probability that there are exactly 7 liberal senators?

EXAMPLE: In the Senate, there are 53 liberals and 47 conservatives. A committee is composed randomly of 9 senators. We know that the committee does not contain more than 3 conservative senators. What is the probability that there are exactly 7 liberal senators?

Solution: There are $\binom{53}{7}$ ways to choose 7 out of 53 liberals and $\binom{47}{2}$ ways to choose 2 out of 47 conservatives. Therefore, by the Basic Principle of Counting, there are a total of $\binom{53}{7}\binom{47}{2}$ possible committees with exactly 7 liberal senators. Similarly, there are

$$\binom{47}{0}\binom{53}{9}+\binom{47}{1}\binom{53}{8}+\binom{47}{2}\binom{53}{7}+\binom{47}{3}\binom{53}{6}$$

ways to choose $0, 1, 2,$ or 3 conservative senators. Therefore the probability that there are exactly 7 liberal senators is

$$P(A)=\frac{\binom{53}{7}\binom{47}{2}}{\binom{47}{0}\binom{53}{9}+\binom{47}{1}\binom{53}{8}+\binom{47}{2}\binom{53}{7}+\binom{47}{3}\binom{53}{6}}\approx 0.28485$$

The major shortcoming of the classical probability concept is its limited applicability, for there are many situations in which the various probabilities cannot all be regarded as equally likely. For example, if we are concerned with the question of whether it will rain the next day, whether a missile launching will be a success, whether a newly designed engine will function for at least 1000 hours, or whether a certain candidate will win an election.

Among the various probability concepts, most widely held is the **frequency interpretation:**

> **The probability of an event (happening or outcome) is the proportion of the time that events of the same kind will occur in the long run.**

For example, if we say that the probability is 0.78 that a jet from New York to Boston will arrive on time, we mean that such flights arrive on time 78% of the time.

In accordance with the frequency interpretation of probability, we estimate the probability of an event by observing what fraction of the time similar events have occurred in the past.

EXAMPLE: If records show that 294 of 300 ceramic insulators tested were able to withstand a certain thermal shock, what is the probability that any one untested insulator will be able to withstand the thermal shock?

Solution: Among the tested insulators, $294/300 = 0.98$ were able to withstand the thermal shock, and we use this frequency as an estimate of the probability.

When probabilities are estimated in this way, it is only reasonable to ask whether the estimates are any good. To answer this question, let us refer to an important theorem called the **law of large numbers**. Informally, this theorem may be stated as follows:

> **If a situation, trial, or experiment is repeated again and again, the proportion of successes will tend to approach the probability that any one outcome will be a success.**

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 330 165 55 11 1

1 12 66 220 495 792 924 792 495 220 66 12 1

1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1

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