Final Examination

V63.0121: Calculus I

May 8, 2009

PLEASE CHECK YOUR SECTION. IF YOU ARE IN SECTION 10, PLEASE ALSO CHECK YOUR RECITATION.

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<thead>
<tr>
<th>Section</th>
<th>Meets</th>
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<tr>
<td>001</td>
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<td>Ning Jiang</td>
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<td>Juliana Faus da Silva Dias</td>
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<td>003</td>
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<td>015</td>
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<td>SILV 504</td>
<td>Michael Damron</td>
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PLEASE READ THE FOLLOWING INFORMATION.

- This is a 110-minute exam. Calculators, formula sheets, and other aids are not allowed.

- Read the instructions of every problem. Unless otherwise stated, show all of your work. Full credit may not be given for an answer alone.

- You may use the backs of the pages or the extra pages for scratch work. Do not unstaple or remove pages as they can be lost in the grading process.

- Please do not put your name on any page besides the first page.

Good luck!
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<tr>
<th>Problem Number</th>
<th>Possible Points</th>
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1. (10 Points) Determine whether the following statements are true or false. Please fill in the circle. No justification is necessary.

(i) The limit \( \lim_{x \to 2} (x^2 + x - 2) = 10 \)

(ii) The function defined by

\[
    f(x) = \begin{cases} 
        1 - x & x < 1 \\ 
        x^2 - 2x + 1 & x \geq 1 
    \end{cases}
\]

is continuous at 1.

(iii) The function defined by

\[
    f(x) = \begin{cases} 
        -2x & x < -1 \\ 
        x^2 & x \geq -1 
    \end{cases}
\]

is differentiable at \(-1\).

(iv) The limit \( \lim_{x \to \infty} \frac{x^3}{e^x} = 0 \).

(v) The limit \( \lim_{y \to 0} \frac{\sin 3y}{2y} \) does not exist.
2. (20 Points) Find following derivatives. *Show your work. You need not simplify your answers.*

(i) \( \frac{d}{dx} \cos(2x) \sqrt{x^2 + 1} \)

(ii) \( \frac{dy}{dt} \), where \( y = \sin^{-1}(te') \)

*Note.* \( \sin^{-1} \) is the inverse of sine, also known as arcsin, but this is *not* the same as the reciprocal of sine.

(iii) \( f'(x) \), where \( f(x) = \log_5(2x) \)

*Hint.* You can use the identity \( \log_5 x = \frac{\ln x}{\ln 5} \).
(iv) $\frac{dy}{dx}$ along the curve $x^2 + \sin^2 y = 1$ Your answer can have $x$ and $y$ in it.

(v) $\frac{dy}{dx}$, where $y = x^{2x}$
3. (10 Points) Let \( f(x) = \frac{2}{3}x^3 + x^2 + \frac{16}{3} \).

(a) Let \( a \) be a point in the domain of \( f \). Explain the relationship between the derivative \( f'(a) \) and the tangent line to the graph at \( (a, f(a)) \).

(b) Use the definition of the derivative to write down a limit equal to \( f'(1) \). You need not calculate the limit.

(c) Show that \( f'(1) = 4 \). Shortcuts as in Problem 1 are allowed here.

(d) Find the equation for the tangent line to the graph of \( f \) at the point \( P = (1, 7) \). Express your answer in slope-intercept form.

(e) Use a linear approximation to \( f \) at 1 to estimate \( f(1.5) \).
4. (10 Points) We would like to investigate the function \( f(x) = 3x^{1/3} - x + 2 \). For reference,

\[
f'(x) = x^{-2/3} - 1 \quad f''(x) = -\frac{2}{3}x^{-5/3}
\]

(a) (2 points) What are the critical points of \( f \) in the interval \((-1, 27)\)? Remember that these are points where the derivative is zero or the function is not differentiable.

(b) (3 points) On which intervals is \( f \) increasing? decreasing?

<table>
<thead>
<tr>
<th>Increasing on</th>
<th>Decreasing on</th>
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(c) (3 points) On which intervals is \( f \) concave up? concave down?

<table>
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<th>Concave up on</th>
<th>Concave down on</th>
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(d) (2 points) Where in the interval \([-1, 27]\) does \( f \) have its absolute maximum? its absolute minimum?

| Maximum at \( x = \) | Minimum at \( x = \) |
5. (10 Points) In each part select the correct lettered answer. Please fill in the circle. No justification is necessary.

(i) Which of these is equal to $\cos \left( \frac{\pi}{2} \right)$?

A. $90^\circ$  
B. 1  
C. 0  
D. $\frac{\sqrt{2}}{2}$  
E. $\frac{1}{2}$

(ii) Which of these is equal to $\ln(e^2)$?

A. $2e$  
B. 2  
C. 4  
D. $\ln 2$  
E. 1

(iii) Which of these is equal to $\arcsin \left( \frac{\sqrt{3}}{2} \right)$?

A. $\frac{\pi}{6}$  
B. $\frac{\pi}{3}$  
C. $\frac{\pi}{4}$  
D. $\frac{1}{\sqrt{1-3/4}}$  
E. $\frac{\pi}{2}$

(iv) Which of these is equal to $\cos^{-1} \left( -\frac{1}{\sqrt{2}} \right)$?

A. $-\frac{\pi}{4}$  
B. $\frac{\pi}{4}$  
C. $\frac{3\pi}{4}$  
D. $-\frac{3\pi}{4}$  
E. $-\frac{1}{\sqrt{1-1/2}}$

(v) Which of these is equal to $\ln(72)$?

I. $\ln(50) + \ln(22)$  
II. $\ln(36) + \ln(2)$  
III. $3\ln(2) + 2\ln(3)$  
IV. $\frac{1}{72}$

A. I, II, and III  
B. IV only  
C. I and III  
D. I only  
E. II and III
6. (10 Points) Consider all rectangles lying in the first quadrant of the \( xy \)-plane with one vertex at \((0,0)\) and one vertex along the curve \( y = \frac{1}{1+4x^2} \). Find the maximum area of such a rectangle. What are the dimensions of the rectangle with maximum area?

*Explain which function you are trying to maximize and over what domain. Also demonstrate that the point you find is truly the absolute maximum.*
7. (10 Points) Let \( f(x) \) be the function graphed below:

Let \( I = \int_a^b f(x) \, dx \). For any \( n \), let \( L_n \) be the Riemann sum for \( I \) using \( n \) intervals (of equal width) and left endpoints, \( R_n \) be the Riemann sum for \( I \) using \( n \) intervals and right endpoints, and \( M_n \) be the Riemann sum for \( I \) using \( n \) subintervals and midpoints.

In each of these, select \( > \) if the quantity in column A is greater, \( < \) if the quantity in column B is greater, \( = \) if the two quantities are the same, and \( ? \) if it is impossible to determine which is greater. Please fill in the circle. No justification is necessary.

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<thead>
<tr>
<th></th>
<th>A</th>
<th>Your answer</th>
<th>B</th>
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<tbody>
<tr>
<td>(i)</td>
<td>( L_2 )</td>
<td>( &lt; ) ( &gt; ) ( = ) ( ? )</td>
<td>( I )</td>
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<tr>
<td>(ii)</td>
<td>( R_2 )</td>
<td>( &lt; ) ( &gt; ) ( = ) ( ? )</td>
<td>( R_4 )</td>
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<td>(iii)</td>
<td>(</td>
<td>M_4 - I</td>
<td>)</td>
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<td>(iv)</td>
<td>( \int_a^d f(x) , dx )</td>
<td>( &lt; ) ( &gt; ) ( = ) ( ? )</td>
<td>( \int_a^c f(x) , dx + \int_c^d f(x) , dx )</td>
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<td>(v)</td>
<td>( \int_a^d f(x) , dx )</td>
<td>( &lt; ) ( &gt; ) ( = ) ( ? )</td>
<td>( \int_a^c f(x) , dx )</td>
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8. (10 Points) For \( x \) in \([-1, \infty)\), let
\[
g(x) = \int_{0}^{x} \sqrt{1 + t^3} \, dt
\]

Note. It is **mathematically impossible** to evaluate this integral in elementary terms, so please do not try. That is not the point of this problem.

(i) (2 points) State the First Fundamental Theorem of Calculus (the first part is the one which seems relevant to this problem).

(ii) (3 points) Find \( g'(1) \).

(iii) (2 points) On which intervals is \( g \) increasing?

(iv) (3 points) Find the derivative of \( h(x) = g(e^x) \).
9. (10 Points) Evaluate the following integrals. In the case of an definite integral, your answer should be a real number. In the case of an indefinite integral, your answer should be the most general antiderivative as a function of the original variable.

(i) (3 points) \[ \int (2x + 1)^2 \, dx \]

(ii) (3 points) \[ \int_0^1 xe^{-x^2} \, dx \]

(iii) (4 points) \[ \int \frac{2 \cos x}{1 + 4 \sin^2 x} \, dx \]