Solutions to Final Examination

V63.0121: Calculus I

May 8, 2009
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<tr>
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1. (10 Points) Determine whether the following statements are true or false. Please fill in the circle. No justification is necessary.

(i) The limit \( \lim_{x \to 2} (x^2 + x - 2) = 10 \)

\[ \text{T} \] \[ \text{F} \]

Solution. By Direct Substitution, the limit is 4.

(ii) The function defined by

\[ f(x) = \begin{cases} 
1 - x & x < 1 \\
-x^2 - 2x + 1 & x \geq 1 
\end{cases} \]

is continuous at 1.

\[ \text{F} \]

Solution. The limit from the left is 0, as is the limit from the right. So the limit is 0. Since \( f(1) = 0 \), the function is continuous.

(iii) The function defined by

\[ f(x) = \begin{cases} 
-2x & x < -1 \\
x^2 & x \geq -1 
\end{cases} \]

is differentiable at \(-1\).

\[ \text{T} \]

Solution. This function is not continuous at \(-1\).

(iv) The limit \( \lim_{x \to \infty} \frac{x^3}{e^x} = 0 \).

\[ \text{F} \]

Solution. Exponential functions grow faster than all power functions.

(v) The limit \( \lim_{y \to 0} \frac{\sin 3y}{2y} \) does not exist.

\[ \text{T} \]

Solution. The limit is \( \frac{3}{2} \).
2. (20 Points) Find following derivatives. Show your work. You need not simplify your answers.

(i) \( \frac{d}{dx} \frac{\cos(2x)}{\sqrt{x^2+1}} \)

Solution. We have

\[
\frac{d}{dx} \frac{\cos(2x)}{\sqrt{x^2+1}} = \frac{\sqrt{x^2+1} \cdot (-2 \sin(2x)) - \cos(2x) \cdot \frac{1}{2\sqrt{x^2+1}} \cdot (2x)}{x^2 + 1}
\]

\[
= -\frac{2 \sin(2x)}{\sqrt{x^2+1}} - \frac{x \cos(2x)}{(x^2 + 1)^{3/2}}
\]

(ii) \( \frac{dy}{dt} \), where \( y = \sin^{-1}(te^t) \)

Note. \( \sin^{-1} \) is the inverse of sine, also known as arcsin, but this is not the same as the reciprocal of sine.

Solution. We have

\[
\frac{dy}{dt} = \frac{1}{\sqrt{1 - (te^t)^2}} \frac{d}{dt} (te^t) = \frac{te^t + e^t}{\sqrt{1 - t^2e^{2t}}}
\]

(iii) \( f'(x) \), where \( f(x) = \log_5(2x) \)

Hint. You can use the identity \( \log_5 x = \frac{\ln x}{\ln 5} \).

Solution. Following the hint, we have

\[
f'(x) = \frac{1}{\ln 5} \cdot \frac{1}{2x} \cdot 2 = \frac{1}{x\ln 5}
\]
(iv) $\frac{dy}{dx}$ along the curve $x^2 + \sin^2 y = 1$ Your answer can have $x$ and $y$ in it.

**Solution.** Differentiating implicitly, we have

$$2x + 2 \sin y \cos y \frac{dy}{dx} = 0$$

So

$$\frac{dy}{dx} = -\frac{x}{\sin y \cos y}$$

(v) $\frac{dy}{dx}$, where $y = x^{2x}$

**Solution.** Take the logarithm before differentiating: $\ln y = 2x \ln x$, so

$$\frac{1}{y} \frac{dy}{dx} = 2 \ln x + 2x \cdot \frac{1}{x} = 2 \ln x + 2.$$ 

So

$$\frac{dy}{dx} = 2x^{2x} (\ln x + 1)$$
3. (10 Points) Let \( f(x) = \frac{2}{3}x^3 + x^2 + \frac{16}{3} \).

(a) Let \( a \) be a point in the domain of \( f \). Explain the relationship between the derivative \( f'(a) \) and the tangent line to the graph at \((a, f(a))\).

Solution. The derivative \( f'(a) \) is the slope of the line tangent to the graph at \((a, f(a))\).

(b) Use the definition of the derivative to write down a limit equal to \( f'(1) \). You need not calculate the limit.

Solution. We have
\[
  f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{\frac{2}{3}x^3 + x^2 + \frac{16}{3} - 7}{x - 1}
\]

(c) Show that \( f'(1) = 4 \). Shortcuts as in Problem 1 are allowed here.

Solution. This is just the power rule:
\[
  f'(x) = 2x^2 + 2x,
\]
so \( f'(1) = 4 \).

(d) Find the equation for the tangent line to the graph of \( f \) at the point \( P = (1, 7) \). Express your answer in slope-intercept form.

Solution. We know the line goes through the point \( P \) and has slope 4. So it has equation
\[
  y - 7 = 4(x - 1) \implies y = 4x + 3
\]

(e) Use a linear approximation to \( f \) at 1 to estimate \( f(1.5) \).

Solution. The linear approximation to \( f \) at 1 is the tangent line. So we plug in \( x = 1.5 \) to the equation for the tangent line and get
\[
  f(1.5) \approx f(1) + f'(1)(0.5) = 7 + 2 = 9
\]
4. (10 Points) We would like to investigate the function \( f(x) = 3x^{1/3} - x + 2 \). For reference, we have:

\[
\begin{align*}
\frac{d}{dx} f(x) &= x^{-2/3} - 1 \\
\frac{d^2}{dx^2} f(x) &= \frac{-2}{3} x^{-5/3}
\end{align*}
\]

(a) (2 points) What are the critical points of \( f \)? Remember that these are points where the derivative is zero or the function is not differentiable.

**Solution.** The derivative \( \frac{d}{dx} f(x) = 0 \) when \( x = \pm 1 \). The function is not differentiable when \( x = 0 \). So the critical points are 0 and \( \pm 1 \).

(b) (3 points) On which intervals is \( f \) increasing? decreasing?

Increasing on \(-1, 0 \) \( \cup \) \( (0, 1)\) Decreasing on \((-\infty, -1) \cup (1, \infty)\)

**Solution.** Looking at the derivative, we can see it is positive when \( 0 < |x| < 1 \) and negative when \( |x| > 1 \).

(c) (3 points) On which intervals is \( f \) concave up? concave down?

Concave up on \((-\infty, 0)\) Concave down on \((0, \infty)\)

**Solution.** Looking at the second derivative, we see it is positive when \( x < 0 \), and negative when \( x > 0 \).

(d) (2 points) Where in the interval \([-1, 27]\) does \( f \) have its absolute maximum? its absolute minimum?

Maximum at \( x = 1 \) Minimum at \( x = 27 \)

**Solution.** We need to check the endpoints \( x = -1 \) and \( x = 27 \), as well as the critical points \( x = 1 \) and \( x = 0 \). We have:

\[
\begin{align*}
f(-1) &= 0 \\
f(0) &= 2 \\
f(1) &= 4 \\
f(27) &= -16
\end{align*}
\]

---

*The original exam had ‘‘in the interval \((-1, 27)\)’’ in the problem. In that case, the critical point \( x = -1 \) should not be listed since it is not in the interval.*
5. (10 Points) In each part select the correct lettered answer. Please fill in the circle. No justification is necessary.

(i) Which of these is equal to \( \cos \left( \frac{\pi}{2} \right) \)?

- A) \( 90^\circ \)
- B) 1
- C) \( 0 \)
- D) \( \frac{\sqrt{2}}{2} \)
- E) \( \frac{1}{2} \)

(ii) Which of these is equal to \( \ln(e^2) \)?

- A) \( 2e \)
- B) 2
- C) 4
- D) \( \ln 2 \)
- E) 1

(iii) Which of these is equal to \( \arcsin \left( \frac{\sqrt{3}}{2} \right) \)?

- A) \( \frac{\pi}{6} \)
- B) \( \frac{\pi}{3} \)
- C) \( \frac{\pi}{4} \)
- D) \( \frac{1}{\sqrt{1-3/4}} \)
- E) \( \frac{\pi}{2} \)

(iv) Which of these is equal to \( \cos^{-1} \left( -\frac{1}{\sqrt{2}} \right) \)?

- A) \( -\frac{\pi}{4} \)
- B) \( \frac{\pi}{4} \)
- C) \( \frac{3\pi}{4} \)
- D) \( -\frac{3\pi}{4} \)
- E) \( -\frac{1}{\sqrt{1-1/2}} \)

(v) Which of these is equal to \( \ln(72) \)?

- I. \( \ln(50) + \ln(22) \)
- II. \( \ln(36) + \ln(2) \)
- III. \( 3\ln(2) + 2\ln(3) \)
- IV. \( \frac{1}{72} \)

- A) I, II, and III
- B) IV only
- C) I and III
- D) I only
- E) II and III
6. (10 Points) Consider all rectangles lying in the first quadrant of the $xy$-plane with one vertex at $(0, 0)$ and one vertex along the curve $y = \frac{1}{1 + 4x^2}$. Find the maximum area of such a rectangle. What are the dimensions of the rectangle with maximum area?

Explain which function you are trying to maximize and over what domain. Also demonstrate that the point you find is truly the absolute maximum.

Solution. The function we want to maximize is the area of a rectangle with width $x$ and height $y = \frac{1}{1 + 4x^2}$. So the function is

$$A(x) = \frac{x}{1 + 4x^2}$$

on the interval $(0, \infty)$. We have

$$A'(x) = \frac{1 - 4x^2}{(1 + 4x^2)^2},$$

which is positive when $1 - 4x^2 > 0 \iff x < \frac{1}{2}$ and negative when $x > \frac{1}{2}$. So $\frac{1}{2}$ is the unique local maximum, in fact the absolute maximum. The other dimension is $y = \frac{1}{1 + 4(1/2)^2} = \frac{1}{2}$, and so the minimum area is $\frac{1}{4}$.

Dimensions: $\frac{1}{2} \times \frac{1}{2}$

Area: $\frac{1}{4}$
7. (10 Points) Let \( f(x) \) be the function graphed below:

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{function_graph.png}
\end{array}
\end{array}
\]

Let \( I = \int_a^b f(x) \, dx \). For any \( n \), let \( L_n \) be the Riemann sum for \( I \) using \( n \) subintervals (of equal width) and left endpoints, \( R_n \) be the Riemann sum for \( I \) using \( n \) subintervals and right endpoints, and \( M_n \) be the Riemann sum for \( I \) using \( n \) subintervals and midpoints.

In each of these, select \( > \) if the quantity in column A is greater, \( < \) if the quantity in column B is greater, \( = \) if the two quantities are the same, and \( ? \) if it is impossible to determine which is greater. Please fill in the circle. No justification is necessary.

<table>
<thead>
<tr>
<th>A</th>
<th>Your answer</th>
<th>B</th>
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<tbody>
<tr>
<td>(i) ( L_2 )</td>
<td>( &lt; )</td>
<td>( I )</td>
</tr>
<tr>
<td>(ii) ( R_2 )</td>
<td>( &gt; )</td>
<td>( R_4 )</td>
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<tr>
<td>(iii) (</td>
<td>M_4 - I</td>
<td>)</td>
</tr>
<tr>
<td>(iv) ( \int_a^d f(x) , dx )</td>
<td>( &lt; )</td>
<td>( \int_c^c f(x) , dx + \int_d^d f(x) , dx )</td>
</tr>
<tr>
<td>(v) ( \int_a^d f(x) , dx )</td>
<td>( &lt; )</td>
<td>( \int_a^d f(x) , dx )</td>
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8. (10 Points) For $x$ in $[-1, \infty)$, let

$$g(x) = \int_0^x \sqrt{1 + t^3} \, dt$$

**Note.** It is **mathematically impossible** to evaluate this integral in elementary terms, so please do not try. That is not the point of this problem.

(i) (2 points) State the First Fundamental Theorem of Calculus (the first part is the one which seems relevant to this problem).

**Solution.** Let $f$ be an integrable function on $[a, b]$, and define $g(x) = \int_a^x f(t) \, dt$. If $f$ is continuous at $c$ in $(a, b)$, then $g$ is differentiable at $c$, and $g'(c) = f(c)$. ▲

(ii) (3 points) Find $g'(1)$.

**Solution.** According to the FTC, $g'(1) = \sqrt{1 + 1^3} = \sqrt{2}$. ▲

(iii) (2 points) On which intervals is $g$ increasing?

**Solution.** Since $g'(x) = \sqrt{1 + x^3}$, which is positive on $(-1, \infty)$, $g$ is increasing over its entire domain. ▲

(iv) (3 points) Find the derivative of $h(x) = g(e^x)$.

**Solution.** Using the FTC and the chain rule, we get

$$h'(x) = g'(e^x)e^x = e^x \sqrt{1 + e^{3x}}$$

▲
9. (10 Points) Evaluate the following integrals. In the case of a definite integral, your answer should be a real number. In the case of an indefinite integral, your answer should be the most general antiderivative as a function of the original variable.

(i) (3 points) \( \int (2x + 1)^2 \, dx \)

Solution. Let \( u = 2x + 1 \), so \( du = 2 \, dx \) and \( dx = \frac{1}{2} \, du \). So

\[
\int (2x + 1)^2 \, dx = \frac{1}{2} \int u^2 \, du = \frac{1}{2} \frac{u^3}{3} + C = \frac{1}{6} (2x + 1)^3 + C
\]

(ii) (3 points) \( \int_0^1 xe^{-x^2} \, dx \)

Solution. Let \( u = x^2 \), so \( du = 2 \, dx \). Then

\[
\int_0^1 xe^{-x^2} \, dx = \frac{1}{2} \int_0^1 e^{-u} \, du = -\frac{1}{2} e^{-u} \bigg|_{u=0}^{u=1} = \frac{1}{2} \left( 1 - \frac{1}{e} \right)
\]

(iii) (4 points) \( \int \frac{2 \cos x}{1 + 4 \sin^2 x} \, dx \)

Solution. Let \( u = 2 \sin x \), so \( du = 2 \cos x \, dx \). Then

\[
\int \frac{2 \cos x}{1 + 4 \sin^2 x} \, dx = \int \frac{du}{1 + u^2} = \arctan u + C = \arctan(2 \sin x) + C
\]