Calculus I-Review Problems
New York University

Exam is on Sections 2.3-2.6, 2.8, 3.1-3.5, 3.7, 4.1, 4.2, 4.3, 4.4, 4.5, 4.7, 5.1-5.5

1. Compute the following limits:
   (a) \( \lim_{x \to 3^-} e^{2/(x-3)} \)
   Solution:
   
   Let \( t = 2/(x - 3) \). As \( x \to 3^- \), \( t \to -\infty \).
   \( \lim_{x \to 3^-} e^{2/(x-3)} = \lim_{t \to -\infty} e^t = 0 \)

   (b) \( \lim_{x \to 10^-} \ln(100 - x^2) \)
   Solution:
   
   \( \lim_{x \to 10^-} \ln(100 - x^2) = -\infty \) since as \( x \to 10^- \), \( (100 - x^2) \to 0^+ \).

   (c) \( \lim_{x \to \infty} e^{-x} \sin x \)
   Solution:
   
   \( -1 \leq \sin x \leq 1 \) \( \Rightarrow \) \( -e^{-x} \leq e^{-x} \sin x \leq e^{-x} \). Now \( \lim_{x \to \infty} (\pm e^{-x}) = 0 \), so by the Squeeze Theorem,
   \( \lim_{x \to \infty} e^{-x} \sin x = 0. \)

   (d) \( \lim_{x \to \infty} \frac{1 + 2^x}{1 - 2^x} \)
   Solution:
   
   \( \lim_{x \to \infty} \frac{1 + 2^x}{1 - 2^x} = \lim_{x \to \infty} \frac{1/2^x + 1}{1/2^x - 1} = \frac{0 + 1}{0 - 1} = -1 \)

   You can also apply L'Hospital’s,

   (e) \( \lim_{x \to \infty} x^3 e^{-x} \)
   Solution:
   
   \( \lim_{x \to \infty} x^3 e^{-x} = \lim_{x \to \infty} x^3 \frac{e^{-x}}{e^{-x}} = \lim_{x \to \infty} 3x^2 \frac{e^{-x}}{e^{-x}} = \lim_{x \to \infty} 6x \frac{e^{-x}}{e^{-x}} = \lim_{x \to \infty} 6 = 0 \)

   (f) \( \lim_{x \to 1^+} \frac{x}{x - 1} - \frac{1}{\ln x} \)
   Solution:
   
   \( \lim_{x \to 1^+} \left( \frac{x}{x - 1} - \frac{1}{\ln x} \right) = \lim_{x \to 1^+} \left( \frac{x \ln x - x + 1}{(x-1) \ln x} \right) = \lim_{x \to 1^+} \frac{x \cdot (1/x) + \ln x - 1}{(x-1) \cdot (1/x) + \ln x} \)
   \( = \lim_{x \to 1^+} \frac{\ln x}{1 - 1/x + \ln x} = \lim_{x \to 1^+} \frac{1/x}{1/x^2 + 1/x} = \frac{1}{1 + 1} = \frac{1}{2} \)
(g) \( \lim_{x \to \pi/2^-} \tan x \cos x \)

Solution:

\[
y = (\tan x)^{\cos x} \Rightarrow \ln y = \cos x \ln \tan x, \quad \text{so} \\
\lim_{x \to -(\pi/2)^-} \ln y = \lim_{x \to -(\pi/2)^-} \frac{\ln \tan x}{\sec x} = \lim_{x \to -(\pi/2)^-} \frac{(1/\tan x) \sec^2 x}{\sec x \tan x} = \lim_{x \to -(\pi/2)^-} \frac{\sec x}{\sin^2 x} = \frac{0}{1^2} = 0,
\]

so \( \lim_{x \to -(\pi/2)^-} (\tan x)^{\cos x} = \lim_{x \to -(\pi/2)^-} e^{\ln y} = 1 \).

(h) \( \lim_{x \to \infty} x[\ln(x+5) - \ln x] \)

Solution: \( \lim_{x \to \infty} x \ln \frac{x+5}{x} = \lim_{x \to \infty} \frac{\ln(x+5)}{\frac{1}{x}} = \lim_{x \to \infty} \frac{-5}{\frac{1}{x}} = \lim_{x \to \infty} \frac{-5x/x^2(x+5)}{1/x^2} = \lim_{x \to \infty} \frac{5x}{x^3} = -5. \)

2. Calculate \( y' \).

(a) \( y = \frac{2x}{\sqrt{x^2+1}} \)

Solution: \( y' = \frac{2\sqrt{x^2+1} - 2x^2(1)^{-1/2}}{x^2+1} \)

(b) \( y = \frac{1}{\sin(x - \sin x)} \)

Solution:

Using the Reciprocal Rule, \( g(x) = \frac{1}{f(x)} \Rightarrow g'(x) = -\frac{f'(x)}{[f(x)]^2} \), we have \( y = \frac{1}{\sin(x - \sin x)} \Rightarrow \\
y' = -\frac{\cos(x - \sin x)(1 - \cos x)}{\sin^2(x - \sin x)}. \)

(c) \( y = \frac{\sec 2\theta}{1 + \tan 2\theta} \)

Solution:

\[
y' = \frac{(1 + \tan 2\theta)(\sec 2\theta \tan 2\theta \cdot 2) - (\sec 2\theta)(\sec^2 2\theta \cdot 2)}{(1 + \tan 2\theta)^2} = \frac{2 \sec 2\theta [(1 + \tan 2\theta) \tan 2\theta - \sec^2 2\theta]}{(1 + \tan 2\theta)^2} = \frac{2 \sec 2\theta (\tan 2\theta - 1)}{(1 + \tan 2\theta)^2} \left[ 1 + \tan^2 x = \sec^2 x \right]
\]

(d) \( xy^4 + x^2y = x + 3y \)

Solution:

\[
\frac{d}{dx} (xy^4 + x^2y) = \frac{d}{dx} (x + 3y) \Rightarrow x \cdot 4y^3 y' + y^4 \cdot 1 + x^2 \cdot y' + y \cdot 2x = 1 + 3y' \Rightarrow \\
y'(4xy^3 + x^2) = 1 - y^4 - 2xy \Rightarrow y' = \frac{1 - y^4 - 2xy}{4xy^3 + x^2 - 3}
\]

(e) \( \sin(xy) = x^2 - y \)

Solution:

\[
\sin(xy) = x^2 - y \Rightarrow \cos(xy)(y'x + y \cdot 1) = 2x - y' \Rightarrow x \cos(xy)y' + y' = 2x - y \cos(xy) \Rightarrow \\
y'[x \cos(xy) + 1] = 2x - y \cos(xy) \Rightarrow y' = \frac{2x - y \cos(xy)}{x \cos(xy) + 1}
\]
(f) \( y = \ln(x^2e^x) \)
Solution:
\[
y' = \frac{2}{x} + 1
\]

(g) \( y = (\cos x)^x \)
Solution:
\[
y = (\cos x)^x \implies \ln y = \ln(\cos x)^x = x \ln \cos x \implies \frac{y'}{y} = x \cdot \frac{1}{\cos x} \cdot (-\sin x) + \ln \cos x \cdot 1 \implies \]
\[
y' = (\cos x)^x (\ln x - x \tan x)
\]

(h) \( y = \ln(\frac{1}{x}) + \frac{1}{\ln x} \)
Solution:
\[
y = \ln \left(\frac{1}{x}\right) + \frac{1}{\ln x} = \ln x^{-1} + (\ln x)^{-1} = -\ln x + (\ln x)^{-1} \implies y' = -\frac{1}{x} + \frac{1}{x(\ln x)^2} = -\frac{1}{x} - \frac{1}{x(\ln x)^2}
\]

(i) \( y = \arctan(\arcsin \sqrt{x}) \)
Solution:
\[
y = \arctan(\arcsin \sqrt{x}) \implies y' = \frac{1}{1 + (\arcsin \sqrt{x})^2} \cdot \frac{1}{\sqrt{1 - x}} \cdot \frac{1}{2 \sqrt{x}}
\]

(j) \( x e^y = y - 1 \)
Solution:
\[
x e^y = y - 1 \implies e^y + x e^y y' = y' \implies y' = e^y/(1 - x e^y)
\]

(k) \( y = (\tan x)^{1/x} \)
Solution:
\[
y = (\tan x)^{1/x} \implies \ln y = \ln(\tan x)^{1/x} \implies \ln y = \frac{1}{x} \ln \tan x \implies \]
\[
\frac{1}{y} y' = \frac{1}{x} \cdot \frac{1}{\tan x} \cdot \sec^2 x + \ln \tan x \cdot \left(-\frac{1}{x^2}\right) \implies y' = y \left(\frac{\sec^2 x}{x \tan x} - \frac{\ln \tan x}{x^2}\right) \implies \]
\[
y' = (\tan x)^{1/x} \left(\frac{\sec^2 x}{x \tan x} - \frac{\ln \tan x}{x^2}\right) \quad \text{or} \quad y' = (\tan x)^{1/x} \cdot \frac{1}{x} \left(\csc x \sec x - \frac{\ln \tan x}{x}\right)
\]

(l) \( e^{x^2}y = x + y \)
Solution:
\[
\frac{d}{dx} (e^{x^2}y) = \frac{d}{dx} (x + y) \implies e^{x^2} (x^2 y' + y \cdot 2x) = 1 + y' \implies x^2 e^{x^2} y' + 2x ye^{x^2} = 1 + y' \implies \]
\[
x^2 e^{x^2} y' - y' = 1 - 2xy e^{x^2} \implies y' \left(x^2 e^{x^2}y - 1\right) = 1 - 2xy e^{x^2} \implies y' = \frac{1 - 2xy e^{x^2}}{x^2 e^{x^2}y - 1}
\]
3. The graph of \( y = x^3 - 9x^2 - 16x + 1 \) has a slope of 5 at two points. Find the coordinates of the points.

Solution:

\[
y' = 3x^2 - 18x - 16 \\
5 = 3x^2 - 18x - 16 \\
0 = 3x^2 - 18x - 21 \\
0 = x^2 - 6x - 7 \\
0 = (x + 1)(x - 7) \\
x = -1 \text{ or } x = 7.
\]

When \( x = -1 \), \( y = 7 \); when \( x = 7 \), \( y = -209 \).
Thus, the two points are \((-1, 7)\) and \((7, -209)\).

4. Find the equation of the line tangent to \( f(x) \) at \( x = 2 \), if \( f(x) = \frac{x^3}{2} - \frac{4}{3x} \).

Solution:

The slope of the tangent line is the value of the first derivative at \( x = 2 \). Differentiating gives

\[
\frac{d}{dx} \left( \frac{x^3}{2} - \frac{4}{3x} \right) = \frac{d}{dx} \left( \frac{1}{2}x^3 - \frac{4}{3}x^{-1} \right) \\
= \frac{3}{2}x^2 - \frac{4}{3}(-1)x^{-2} \\
= \frac{3}{2}x^2 + \frac{4}{3x^2}.
\]

For \( x = 2 \),

\[
f'(2) = \frac{3}{2}(2)^2 + \frac{4}{3(2)^2} = 6 + \frac{1}{3} = 6.333
\]

and

\[
f(2) = \frac{2^3}{2} - \frac{4}{3(2)} = 4 - \frac{2}{3} = 3.333.
\]

To find the \( y \)-intercept for the tangent line equation at the point \((2, 3.333)\), we substitute in the general equation, \( y = b + mx \), and solve for \( b \).

\[
3.333 = b + 6.333(2) \\
-9.333 = b.
\]

The tangent line has the equation

\[ y = -9.333 + 6.333x. \]

5. (a) Find the slope of the graph of \( f(x) = 1 - e^x \) at the point where it crosses the \( x \)-axis.

(b) Find the equation of the tangent line to the curve at this point.

(c) Find the equation of the line perpendicular to the tangent line at this point. (This is the normal line.)

Solution:

(a) \( f(x) = 1 - e^x \) crosses the \( x \)-axis where \( 0 = 1 - e^x \), which happens when \( e^x = 1 \), so \( x = 0 \). Since \( f'(x) = -e^x \),

\[
f'(0) = -e^0 = -1.
\]

(b) \( y = -x \)

(c) The negative of the reciprocal of \(-1\) is \(1\), so the equation of the normal line is \( y = x \).

6. Suppose \( f \) and \( g \) are differentiable functions with the values shown in the following table. For each of the following functions \( h \), find \( h'(2) \).
(a) \( h(x) = f(x) + g(x) \)

(b) \( h(x) = f(x)g(x) \)

(c) \( h(x) = \frac{f(x)}{g(x)} \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
<th>( f'(x) )</th>
<th>( g'(x) )</th>
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<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>-2</td>
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Solution:

(a) We have \( h'(2) = f'(2) + g'(2) = 5 - 2 = 3 \).

(b) We have \( h'(2) = f'(2)g(2) + f(2)g'(2) = 5(4) + 3(-2) = 14 \).

(c) We have \( h'(2) = \frac{f'(2)g(2) - f(2)g'(2)}{(g(2))^2} = \frac{5(4) - 3(-2)}{4^2} = \frac{26}{16} = \frac{13}{8} \).

7. If you invest \( P \) dollars in a bank account at an annual interest rate of \( r\% \), then after \( t \) years you will have \( B \) dollars, where

\[
B = P \left( 1 + \frac{r}{100} \right)^t.
\]

(a) Find \( dB/dt \), assuming \( P \) and \( r \) are constant. In terms of money, what does \( dB/dt \) represent?

(b) Find \( dB/dr \), assuming \( P \) and \( t \) are constant. In terms of money, what does \( dB/dr \) represent?

Solution:

(a) \( \frac{dB}{dt} = P \left( 1 + \frac{r}{100} \right)^t \ln \left( 1 + \frac{r}{100} \right) \). The expression \( \frac{dB}{dt} \) tells us how fast the amount of money in the bank is changing with respect to time for fixed initial investment \( P \) and interest rate \( r \).

(b) \( \frac{dB}{dr} = P t \left( 1 + \frac{r}{100} \right)^{t-1} \frac{1}{100} \). The expression \( \frac{dB}{dr} \) indicates how fast the amount of money changes with respect to the interest rate \( r \), assuming fixed initial investment \( P \) and time \( t \).

8. Use the following graph to calculate the derivative.

(a) \( h'(2) \) if \( h(x) = (f(x))^3 \)

Solution: Since the point \((2, 5)\) is on the curve, we know \( f(2) = 5 \). The point \((2, 1, 5.3)\) is on the tangent line, so

\[
\text{Slope tangent} = \frac{5.3 - 5}{2.1 - 2} = \frac{0.3}{0.1} = 3.
\]

Thus, \( f'(2) = 3 \).

By the chain rule

\[
h'(2) = 3(f(2))^2 \cdot f'(2) = 3 \cdot 5^2 \cdot 3 = 225.
\]

(b) \( k'(2) \) if \( k(x) = (f(x))^{-1} \)

Solution: Since the point \((2, 5)\) is on the curve, we know \( f(2) = 5 \). The point \((2, 1, 5.3)\) is on the tangent line, so

\[
\text{Slope tangent} = \frac{5.3 - 5}{2.1 - 2} = \frac{0.3}{0.1} = 3.
\]
Thus, $f'(2) = 3$.

By the chain rule

$$k'(2) = -(f(2))^{-2} \cdot f'(2) = -5^{-2} \cdot 3 = -0.12.$$

(c) $g'(5)$ if $g(x) = f^{-1}(x)$

Solution: Since the point $(2, 5)$ is on the curve, we know $f(2) = 5$. The point $(2.1, 5.3)$ is on the tangent line, so

$$\text{Slope tangent} = \frac{5.3 - 5}{2.1 - 2} = 0.3 \approx 3.$$

Thus, $f'(2) = 3$. Since $g$ is the inverse function of $f$ and $f(2) = 5$, we know $f^{-1}(5) = 2$, so $g(5) = 2$.

Differentiating, we have

$$g'(2) = \frac{1}{f'(f(5))} = \frac{1}{f'(2)} = \frac{1}{3}.$$

9. (a) Find $dy/dx$ given that $x^2 + y^2 - 4x + 7y = 15$.

(b) Under what conditions on $x$ and/or $y$ is the tangent line to this curve horizontal? Vertical?

Solution:

(a) By implicit differentiation, we have:

$$2x + 2y \frac{dy}{dx} - 4 + 7 \frac{dy}{dx} = 0$$

$$2y + 7 \frac{dy}{dx} = 4 - 2x$$

$$\frac{dy}{dx} = \frac{4 - 2x}{2y + 7}.$$

(b) The curve has a horizontal tangent line when $dy/dx = 0$, which occurs when $4 - 2x = 0$ or $x = 2$. The curve has a horizontal tangent line at all points where $x = 2$.

The curve has a vertical tangent line when $dy/dx$ is undefined, which occurs when $2y + 7 = 0$ or when $y = -7/2$. The curve has a vertical tangent line at all points where $y = -7/2$.

10. In the following problems find the local linearization of $f(x)$ near $x = 0$ and use this to approximate the value of $a$.

(a) $f(x) = (1 + x)^r, \quad a = (1.2)^{3/5}$

Solution: We have $f(x) = (1 + x)^r$, so $f'(x) = r(1 + x)^{r-1}$. Thus $f'(0) = r$ so the local linearization near $x = 0$ is

$$f(x) \approx f(0) + f'(0)x = 1 + rx.$$

Thus

$$(1 + x)^r \approx 1 + rx \quad \text{for small values of } x.$$

Using the linearization with $r = 3/5$ and $x = 0.2$, we have

$$1.2^{3/5} = 1 + \frac{3}{5} \cdot 0.2 = 1 + 0.12 = 1.12.$$

The actual value is $1.2^{3/5} = 1.117$.

(b) $f(x) = e^{kx}, \quad a = e^{0.3}$

Solution: We have $f(x) = e^{kx}$, so $f'(x) = ke^{kx}$. Thus $f'(0) = k$ so the local linearization near $x = 0$ is

$$f(x) \approx f(0) + f'(0)x = 1 + kx.$$

Thus

$$e^{kx} \approx 1 + kx \quad \text{for small values of } x.$$

Using the linearization with $k = 0.3$ and $x = 1$, we have

$$e^{0.3} \approx 1 + 0.3 = 1.3$$

The actual value is $e^{0.3} = 1.350$. 

(c) \( f(x) = \sqrt{b^2 + x}, \quad a = \sqrt{26} \)

Solution: We have \( f(x) = (b^2 + x)^{1/2} \), so \( f'(x) = (1/2)(b^2 + x)^{-1/2} \). Thus \( f'(0) = 1/(2b) \) so the local linearization near \( x = 0 \) is

\[
f(x) \approx f(0) + f'(0)x = b + \frac{1}{2b}x.
\]

Thus

\[
\sqrt{b^2 + x} \approx b + \frac{1}{2b}x \quad \text{for small values of } x.
\]

Using the linearization with \( b = 5 \) and \( x = 1 \), we have

\[
\sqrt{26} \approx 5 + \frac{1}{10} = 5.1.
\]

The actual value is \( \sqrt{26} = 5.099 \).

11. Let \( y^2 + 4x = 4xy^2 \)

(a) Compute \( \frac{dy}{dx} \).

(b) Find the equation for the tangent line to this curve at \( (1/3, 2) \)

(c) Find the \( x \) and \( y \)-coordinates of all points at which the tangent line to this curve is vertical.

\[\text{Solution:}\]
Differentiating the equation with respect to \( x \), we have

\[
2y \frac{dy}{dx} + 4 = 4y^2 + 8xy \frac{dy}{dx}.
\]

Gathering terms involving \( \frac{dy}{dx} \) to one side, the equation becomes

\[
2y \frac{dy}{dx} - 8xy \frac{dy}{dx} = 4y^2 - 4
\]

which gives the solution

\[
\frac{dy}{dx} = \frac{4y^2 - 4}{2y - 8xy}.
\]

\[\text{Solution:}\]
The slope is

\[
\left. \frac{dy}{dx} \right|_{(\frac{1}{3}, 2)} = \frac{4 \cdot 2^2 - 4}{2 \cdot 2 - 8 \cdot \frac{1}{3} \cdot 2} = -9,
\]

so by the point-slope formula, the equation is

\[
y = -9x + 5.
\]

\[\text{Solution:}\]
The slope is undefined at these points, so we must have \( 2y - 8xy = 0 \). Factoring out a \( 2y \) we get

\[
2y(1 - 4x) = 0
\]

which gives the solutions \( y = 0 \) or \( x = \frac{1}{4} \). Plugging into the equation for the implicit function, \( y = 0 \) gives the point \((0, 0)\). However, when we plug in \( x = \frac{1}{4} \), we get the equation \( y^2 + 1 = y^2 \), which has no solutions. Therefore, \((0, 0)\) is the only point at which the tangent line is vertical.
12. Find the local linearization \( L(x) \) of the function \( f(x) = (1 + x)^k \) near \( x = 0 \), where \( k \) is a positive constant. Suppose you want to use \( L(x) \) to find an approximation of the number \( \sqrt{1.1} \). What number should \( k \) be, and what number should \( x \) be? Approximate \( \sqrt{1.1} \).

**Solution:** The derivative is \( f'(x) = k(1 + x)^{k-1} \), so the slope of the tangent line at \( x = 0 \) is

\[ f'(0) = k. \]

Since \( f(0) = 1^k = 1 \), the tangent line passes through the point \((0, 1)\). Therefore, the point-slope formula shows that the equation of the tangent line is

\[ y = kx + 1. \]

**Solution:** If \( k = \frac{1}{2} \) and \( x = 0.1 \), then \( f(0.1) = \sqrt{1.1} \), so \( L(1.1) \) gives an approximation of \( \sqrt{1.1} \).

**Solution:** If \( k \) and \( x \) are as above, then \( \sqrt{1.1} \approx L(0.1) = 1.05. \)

13. Use a linear approximation to estimate the given number

(a) \( (8.06)^{2/3} \)

Solution:

To estimate \( (8.06)^{2/3} \), we’ll find the linearization of \( f(x) = x^{2/3} \) at \( a = 8 \). Since \( f'(x) = \frac{2}{3}x^{-1/3} = 2/3 \sqrt[3]{1/3} \), \( f(8) = 4 \), and \( f'(8) = \frac{1}{3} \), we have \( L(x) = 4 + \frac{1}{3}(x - 8) = \frac{1}{3}x + \frac{4}{3} \). Thus, \( x^{2/3} \approx \frac{1}{3}x + \frac{4}{3} \) when \( x \) is near 8, so

\( (8.06)^{2/3} \approx \frac{1}{3}(8.06) + \frac{4}{3} = \frac{12.98}{3} = 4.03. \)

(b) \( \sqrt{99.8} \)

Solution:

For the sake of variety let’s solve this using differentials. The problem could be very well solved by writing the tangent line approximation as above.

\[ y = f(x) = \sqrt{x} \Rightarrow dy = \frac{1}{2 \sqrt{x}} \, dx. \]

When \( x = 100 \) and \( dx = -0.2 \), \( dy = \frac{1}{2 \sqrt{100}}(-0.2) = -0.01 \), so

\( \sqrt{99.8} = f'(99.8) \approx f(100) + dy = 10 - 0.01 = 9.99. \)

14. The edge of a cube was found to be 30 cm with a possible error in measurement of 0.1 cm. Estimate the maximum possible error in computing

(a) the volume of the cube

(b) the surface area of the cube.

Solution:
15. For what values of \( r \) does the function \( y = e^{rx} \) satisfies the differential equation \( y'' + 5y' - 6y = 0 \)?

Solution:

\[
y = e^{rx} \quad \Rightarrow \quad y' = re^{rx} \quad \Rightarrow \quad y'' = r^2 e^{rx}, \text{ so}
\]

\[
y'' + 5y' - 6y = r^2 e^{rx} + 5re^{rx} - 6e^{rx} = e^{rx}(r^2 + 5r - 6) = e^{rx}(r + 6)(r - 1) = 0 \quad \Rightarrow \quad (r + 6)(r - 1) = 0 \quad \Rightarrow \quad r = 1 \text{ or } -6.
\]

16. If \( f(x) = 3 + x + e^x \), find \((f^{-1})'(4)\).

Solution:

We use Theorem 3.2.7. Note that \( f(0) = 3 + 0 + e^0 = 4 \), so \( f^{-1}(4) = 0 \). Also \( f'(x) = 1 + e^x \). Therefore,

\[
(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(0)} = \frac{1}{1 + e^0} = \frac{1}{2}.
\]

17. A bacteria culture contains 200 cells initially and grows at a rate proportional to its size. After half an hour the population has increased to 360 cells.

(a) Find the number of bacteria after \( t \) hours.
(b) Find the bacteria after 4 hours.
(c) Find the rate of growth after 4 hours.
(d) When will the population reach 10,000.

Solution:

(a) \( y(t) = y(0)e^{kt} = 200e^{kt} \Rightarrow y(0.5) = 200e^{0.5k} = 360 \Rightarrow e^{0.5k} = 1.8 \Rightarrow 0.5k = \ln 1.8 \Rightarrow k = 2 \ln 1.8 = \ln(1.8)^2 = \ln 3.24 \Rightarrow y(t) = 200e^{\ln(3.24)t} = 200(3.24)^t
\]

(b) \( y(4) = 200(3.24)^4 \approx 22,040 \) bacteria

(c) \( y'(t) = 200(3.24)^t \cdot \ln 3.24 \), so \( y'(4) = 200(3.24)^4 \cdot \ln 3.24 \approx 25,910 \) bacteria per hour

(d) \( 200(3.24)^t = 10,000 \Rightarrow (3.24)^t = 50 \Rightarrow t \ln 3.24 = \ln 50 \Rightarrow t = \ln 50 / \ln 3.24 \approx 3.33 \text{ hours} \)
18. At what point on the curve $y = [\ln(x + 4)]^2$ is the tangent horizontal?

Solution:

$$y = [\ln(x + 4)]^2 \quad \Rightarrow \quad y' = 2[\ln(x + 4)] \cdot \frac{1}{x + 4} \cdot 1 = 2 \frac{\ln(x + 4)}{x + 4} \quad \text{and} \quad y' = 0 \iff \ln(x + 4) = 0 \iff x + 4 = e^0 \iff x + 4 = 1 \iff x = -3, \text{so the tangent is horizontal at the point } (-3, 0).$$

19. (a) Find an equation of the tangent to the curve $y = e^x$ that is parallel to the line $x - 4y = 1$.

(b) Find an equation of the tangent to the curve $y = e^x$ that passes through the origin.

Solution:

(a) The line $x - 4y = 1$ has slope $\frac{1}{4}$. A tangent to $y = e^x$ has slope $\frac{1}{4}$ when $y' = e^x = \frac{1}{4} \Rightarrow x = \ln \frac{1}{4} = -\ln 4$.

Since $y = e^x$, the $y$-coordinate is $\frac{1}{4}$ and the point of tangency is $(-\ln 4, \frac{1}{4})$. Thus, an equation of the tangent line is $y - \frac{1}{4} = \frac{1}{4}(x + \ln 4)$ or $y = \frac{1}{4}x + \frac{1}{4}(\ln 4 + 1)$.

(b) The slope of the tangent at the point $(a, e^a)$ is $\frac{dy}{dx} \bigg|_{x=a} = e^a$. Thus, an equation of the tangent line is $y - e^a = e^a(x - a)$. We substitute $x = 0, y = 0$ into this equation, since we want the line to pass through the origin:

$$0 - e^a = e^a(0 - a) \iff -e^a = e^a(-a) \iff a = 1. \text{ So an equation of the tangent line at the point } (a, e^a) = (1, e) \text{ is } y - e = e(x - 1) \text{ or } y = ex.$$

20. Find the equation of the tangent line to the curve $y = 3 \arccos(x/2)$ at $(1, \pi)$.

Solution: The slope of the line is $y'$ evaluated at 1. $y' = -\frac{1}{\sqrt{1-(x/2)^2}} \cdot 1/2$. Hence plugging 1 for $x$ we get $f'(1) = -\sqrt{3}$. So the equation of the tangent line is $y = -\sqrt{3}(x - 1) + \pi$.

21. Bismuth-210 has a half-life of 5 days. A sample population after 1 day is decaying at a rate of 2 mg/day. Find the initial mass of the sample. Write a model describing the population of the Bismuth-210 sample.

Solution: $1/2 = e^{5k}$. Hence $k = 0.2 \ln(0.5)$. So $B = B(0)e^{0.2 \ln(0.5)t}$. Since the decay rate at $t = 1$ is 1 mg/day we know that $\frac{dB}{dt} \bigg|_{t=1} = 2$. Taking derivative of $B$ with respect to $t$ we get

$$\frac{dB}{dt} = B(0)0.2 \ln(0.5)e^{0.2 \ln(0.5)t}.$$ 

Plugging $t = 1$ we get

$$2 = B(0)0.2 \ln(0.5)e^{0.2 \ln(0.5)}.$$ 

So $B(0) = \frac{2}{0.2 \ln(0.5)e^{0.2 \ln(0.5)}}$

22. Find the critical points of the function.

(a) $f(x) = x^3 + 3x^2 - 24x$
(b) $h(p) = \frac{p-1}{p^2+4}$
(c) $f(\theta) = 2\cos\theta + \sin^2\theta$
(d) $f(x) = x \ln x$
(e) $f(x) = xe^{2x}$
Solution:

23. Find the local max/min and the absolute max/min of $f$ on the given interval.

(a) $f(x) = x^4 - 2x^2 + 3$, $[-2, 3]$
(b) $f(x) = \frac{x^2}{x^2 + 4}$, $[0, 3]$
(c) $f(x) = x - 2 \cos x$, $[-\pi, \pi]$
(d) $f(x) = x - \ln x$, $[1/2, 2]$

Solution:

(i) $f(x) = x^3 + 3x^2 - 24x \Rightarrow f'(x) = 3x^2 + 6x - 24 = 3(x^2 + 2x - 8)$.

$f'(x) = 0 \Rightarrow 3(x + 4)(x - 2) = 0 \Rightarrow x = -4, 2$. These are the only critical numbers.

(ii) $h(p) = \frac{p - 1}{p^2 + 4} \Rightarrow h'(p) = \frac{(p^2 + 4)(1) - (p - 1)(2p)}{(p^2 + 4)^2} = \frac{p^2 + 4 - 2p^2 - 2p}{(p^2 + 4)^2} = \frac{-p^2 + 2p + 4}{(p^2 + 4)^2}$.

$h'(p) = 0 \Rightarrow p = \frac{-2 \pm \sqrt{1 + 16}}{-2} = 1 \pm \sqrt{5}$. The critical numbers are $1 \pm \sqrt{5}$. [If $h'(p)$ exists for all real numbers.]

(iii) $f(\theta) = 2\cos \theta + \sin^2 \theta \Rightarrow f'(\theta) = -2\sin \theta + 2\sin \theta \cos \theta$. $f'(\theta) = 0 \Rightarrow 2\sin \theta (\cos \theta - 1) = 0 \Rightarrow \sin \theta = 0$ or $\cos \theta = 1 \Rightarrow \theta = n\pi$ (n is an integer) or $\theta = 2n\pi$. The solutions $\theta = n\pi$ include the solutions $\theta = 2n\pi$, so the critical numbers are $\theta = n\pi$.

(iv) $f(x) = x\ln x \Rightarrow f'(x) = x(\frac{1}{x}) + (\ln x) \cdot 1 = \ln x + 1$. $f'(x) = 0 \Leftrightarrow \ln x = -1 \Leftrightarrow x = e^{-1} = 1/e$. Therefore, the only critical number is $x = 1/e$.

24. Show that 5 is a critical number of the function $g(x) = 2 + (x - 5)^3$ but $g$ does not have a local extreme value at 5.

Solution:
25. Consider the piecewise linear function $f(x)$ graphed below.

For each function $g(x)$, find the value of $g'(x)$:

(a) $g(x) = \sin(f(x)^3)$

(b) $g(x) = \frac{f(x^2)}{x}$

(c) $g(x) = \ln(f(x)) + f(2)$

(d) $g(x) = f^{-1}(x)$

a. [4 points] $g(x) = \sin(f(x)^3)$

Solution:

\[
g'(x) = \cos(f(x)^3) \cdot 3f(x^2) \cdot f'(x)
\]
\[
g'(3) = \cos(7^3) \cdot 3 \cdot 7^2 \cdot (-1) = 124.0442.
\]

b. [4 points] $g(x) = \frac{f(x^2)}{x}$

Solution:

\[
g'(x) = \frac{x \cdot f'(x^2) \cdot 2x - f(x^2)}{x^2}
\]
\[
g'(3) = \frac{3(-3)6 - (-3)}{9} = -5.667.
\]

c. [4 points] $g(x) = \ln(f(x)) + f(2)$

Solution:

\[
g'(x) = \frac{1}{f(x)}f'(x) + 0
\]
\[
g'(3) = \frac{1}{7} \cdot (-1) = -\frac{1}{7}.
\]

d. [4 points] $g(x) = f^{-1}(x)$

Solution:

\[
g'(x) = \frac{1}{f'(f^{-1}(x))}
\]
\[
g'(3) = \frac{1}{f'(6)} = -\frac{2}{3}.
\]
26. Verify that \( f(x) = x^3 + x - 1 \) satisfies the hypothesis of the Mean Value Theorem on the interval \([0, 2]\). Then find all numbers \( c \) that satisfy the conclusion of the Mean Value Theorem.

Solution:

\[
f(x) = x^3 + x - 1, \quad [0, 2], \quad f \text{ is continuous on } [0, 2] \text{ and differentiable on } (0, 2).
\]

\[
f'(c) = \frac{f(2) - f(0)}{2 - 0}
\]

\[
3c^2 + 1 = \frac{9 - (-1)}{2} \iff 3c^2 = 5 - 1 \iff c^2 = \frac{4}{3} \iff c = \pm \frac{2}{\sqrt{3}}, \text{ but only } \frac{2}{\sqrt{3}} \text{ is in } (0, 2).
\]

27. Does there exist a function \( f \) such that \( f(0) = -1, f(2) = 4 \) and \( f'(x) \leq 2 \) for all \( x \)?

Solution:

Suppose that such a function \( f \) exists. By the Mean Value Theorem there is a number \( 0 < c < 2 \) with

\[
f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{5}{2}
\]

But this is impossible since \( f'(x) \leq 2 < \frac{5}{2} \) for all \( x \), so no such function can exist.

28. The graph of \( f \) is given below.

\[
\begin{align*}
f'(x) & \quad x \\
\hline
-1 & \uparrow \\
1 & \downarrow 
\end{align*}
\]

(a) Over what intervals is \( f \) increasing? decreasing?
(b) Does \( f \) have a local maxima or minima, if so which and where?

Solution:

(a) Decreasing for \( x < -1 \), increasing for \(-1 < x < 0 \), decreasing for \( 0 < x < 1 \), and increasing for \( x > 1 \).
(b) \( f(-1) \) and \( f(1) \) are local minima, \( f(0) \) is a local maximum.

29. (a) Find the intervals of increase or decrease.
(b) Find the local maximum and minimum values.
(c) Find the intervals of concavity and the inflection points.
(d) Use the information from parts (a)(c) to sketch the graph.

I. \( h(x) = (x^2 - 1)^3 \)

Solution:
(a) \( h(x) = (x^2 - 1)^3 \Rightarrow h'(x) = 6x(x^2 - 1)^2 \geq 0 \Leftrightarrow x > 0 \ (x \neq 1), \) so \( h \) is increasing on \((0, \infty)\) and decreasing on \((-\infty, 0)\).

(b) \( h(0) = -1 \) is a local minimum value.

c) \( h''(x) = 6(x^2 - 1)^2 + 24x^2(x^2 - 1) = 6(x^2 - 1)(5x^2 - 1) \). The roots \( \pm 1 \) and \( \pm \frac{1}{\sqrt{5}} \) divide \( \mathbb{R} \) into five intervals.

<table>
<thead>
<tr>
<th>Interval</th>
<th>( x^2 - 1 )</th>
<th>( 6x^2 - 1 )</th>
<th>( h''(x) )</th>
<th>Concavity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x &lt; -1 )</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>upward</td>
</tr>
<tr>
<td>( -1 &lt; x &lt; -\frac{1}{\sqrt{5}} )</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>downward</td>
</tr>
<tr>
<td>( -\frac{1}{\sqrt{5}} &lt; x &lt; \frac{1}{\sqrt{5}} )</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>upward</td>
</tr>
<tr>
<td>( \frac{1}{\sqrt{5}} &lt; x &lt; 1 )</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>downward</td>
</tr>
<tr>
<td>( x &gt; 1 )</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>upward</td>
</tr>
</tbody>
</table>

From the table, we see that \( h \) is CU on \((-\infty, -1), \left( -\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right) \) and \((1, \infty)\), and CD on \((-1, -\frac{1}{\sqrt{5}}) \) and \( \left( \frac{1}{\sqrt{5}}, 1 \right) \). There are IPs at \((\pm 1, 0) \) and \( \left( \pm \frac{1}{\sqrt{5}}, \frac{-4}{105} \right) \).

II. \( \ln(x^4 + 27) \)
Solution:

(a) \( f(x) = \ln(x^4 + 27) \Rightarrow f'(x) = \frac{4x^3}{x^4 + 27} \). \( f'(x) > 0 \) if \( x > 0 \) and \( f'(x) < 0 \) if \( x < 0 \), so \( f \) is increasing on \((0, \infty)\) and \( f \) is decreasing on \((-\infty, 0)\).

(b) \( f(0) = \ln 27 \approx 3.3 \) is a local minimum value.

c) \( f''(x) = \frac{(x^4 + 27)(12x^2) - 4x^3(4x^3)}{(x^4 + 27)^2} = \frac{4x^2[3(x^4 + 27) - 4x^4]}{(x^4 + 27)^2} \)
\[ = \frac{4x^2(81 - x^4)}{(x^4 + 27)^2} = \frac{-4x^2(x^2 + 9)(x + 3)(x - 3)}{(x^4 + 27)^2} \]
\( f''(x) > 0 \) if \( -3 < x < 0 \) and \( 0 < x < 3 \), and \( f''(x) < 0 \) if \( x < -3 \) or \( x > 3 \).

Thus, \( f \) is CU on \((-3, 0) \) and \((0, 3) \) [hence on \((-3, 3)\)] and \( f \) is CD on \((-\infty, -3) \) and \((3, \infty) \). There are IPs at \((\pm 3, \ln 108) \approx (\pm 3, 4.68) \).

III. \( \frac{e^x}{1 + e^x} \)
Solution:
\[ f(x) = \frac{e^x}{1 + e^x} \] has domain \( \mathbb{R} \).

(a) \[ \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{e^x}{e^x + 1} = \lim_{x \to \infty} \frac{1}{1 + e^{-x}} = 1, \] so \( y = 1 \) is a HA.

\[ \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{e^x}{e^x + 1} = 0, \] so \( y = 0 \) is a HA. No VA.

(b) \[ f''(x) = \frac{(1 + e^x)^2 e^x - e^x \cdot e^x}{(1 + e^x)^2} = \frac{e^x}{(1 + e^x)^2} > 0 \] for all \( x \). Thus, \( f \) is increasing on \( \mathbb{R} \).

(c) No local maximum or minimum

(d) \[ f''(x) = \frac{(1 + e^x)^2 e^x - e^x \cdot e^x}{(1 + e^x)^2} = \frac{e^x}{(1 + e^x)^2} \]

\[ f''(x) > 0 \ \Leftrightarrow \ 1 - e^x > 0 \ \Leftrightarrow \ x < 0, \] so \( f \) is CU on \( (-\infty, 0) \) and CD on \( (0, \infty) \). There is an IP at \( (0, \frac{1}{2}) \).

IV. \( x\sqrt{5} - x \)

Solution:

\[ y = f(x) = x\sqrt{5} - x \quad \text{A. The domain is } \{x \mid 5 - x \geq 0\} = (-\infty, 5] \quad \text{B. } y\text{-intercept: } f(0) = 0; \]

\( x \)-intercepts: \( f(x) = 0 \ \Leftrightarrow \ x = 0, 5 \)

C. No symmetry

D. No asymptote

E. \[ f'(x) = x \cdot \frac{1}{2} (5 - x)^{-1/2} (-1) + (5 - x)^{1/2} \cdot 1 = \frac{1}{2}(5 - x)^{-1/2}[-x + 2(5 - x)] = \frac{10 - 3x}{2\sqrt{5-x}} > 0 \ \Leftrightarrow \ x < \frac{10}{3}, \]

so \( f \) is increasing on \( (-\infty, \frac{10}{3}) \) and decreasing on \( (\frac{10}{3}, 5) \).

F. Local maximum value \( f\left(\frac{10}{3}\right) = \frac{10}{3} \sqrt{5} \approx 4.3 \); no local minimum

G. \[ f''(x) = \frac{2(5 - x)^{1/2}(-3) - (10 - 3x) \cdot 2\left(\frac{1}{2}\right)(5 - x)^{-1/2}(-1)}{(2\sqrt{5-x})^2} \]

\[ = \frac{(5 - x)^{-1/2}[-6(5-x) + 10 - 3x]}{4(5-x)} = \frac{3x - 20}{4(5 - x)^{3/2}} \]

\[ f''(x) < 0 \text{ for } x < 5, \] so \( f \) is CD on \( (-\infty, 5) \). No IP

30. Find values of \( a \) and \( b \) so that the function \( f(x) = x^2 + ax + b \) has a local minimum at the point \( (6, -5) \).

Solution: First, we wish to have \( f'(6) = 0 \), since \( f(6) \) should be a local minimum:

\[ f'(x) = 2x + a = 0 \]

\[ x = -\frac{a}{2} = 6 \]

\[ a = -12. \]

Next, we need to have \( f(6) = -5 \), since the point \( (6, -5) \) is on the graph of \( f(x) \). We can substitute \( a = -12 \) into our equation for \( f(x) \) and solve for \( b \):

\[ f(x) = x^2 - 12x + b \]

\[ f(6) = 36 - 72 + b = -5 \]

\[ b = 31. \]

Thus, \( f(x) = x^2 - 12x + 31 \).
31. Find the global maximum and minimum of the function \( f(x) = \frac{x + 1}{x^2 + 3} \) on the interval \(-1 \leq x \leq 2\).

Solution: Since the denominator is never 0, we have that \( f(x) = \frac{x + 1}{(x^2 + 3)} \) is continuous. As the interval \(-1 \leq x \leq 2\) is closed, there must be a global maximum and minimum. The candidates are critical points in the interval and endpoints. Since there are no points where \( f'(x) \) is undefined, we solve \( f'(x) = 0 \) to find all the critical points:

\[
f'(x) = \frac{(x^2 + 3) - (x + 1)(2x)}{(x^2 + 3)^2} = \frac{-x^2 - 2x + 3}{(x^2 + 3)^2} = 0.
\]

Thus \(-x^2 - 2x + 3 = -(x + 3)(x - 1) = 0\), so \(x = -3\) and \(x = 1\) are the critical points; only \(x = 1\) is in the interval. We then compare the values of \(f\) at the critical points and the endpoints:

\[
f(-1) = 0, \quad f(1) = \frac{1}{2}, \quad f(2) = \frac{3}{7}.
\]

Thus the global maximum is \(1/2\) at \(x = 1\), and the global minimum is \(0\) at \(x = -1\).

32. If you have 100 feet of fencing and want to enclose a rectangular area up against a long, straight wall, what is the largest area you can enclose?

Solution: Let \(w\) and \(l\) be the width and length, respectively, of the rectangular area you wish to enclose. Then

\[
w + w + l = 100 \text{ feet}
\]

\[
l = 100 - 2w
\]

Area = \(w \cdot l = w(100 - 2w) = 100w - 2w^2\)

To maximize area, we solve \(A' = 0\) to find critical points. This gives \(A' = 100 - 4w = 0\), so \(w = 25\), \(l = 50\). So the area is \(25 \cdot 50 = 1250\) square feet. This is a local maximum by the second derivative test because \(A'' = -4 < 0\). Since the graph of \(A\) is a parabola, the local maximum is in fact a global maximum.

33. A line goes through the origin and a point on the curve \(y = x^2e^{-3x}\), for \(x \geq 0\). Find the maximum slope of such a line. At what \(x\)-value does it occur?

Solution:

\[
\text{Slope} = \frac{y}{x} = \frac{x^2e^{-3x}}{x} = xe^{-3x}.
\]

If the slope has a maximum, it occurs where

\[
\frac{d}{dx} \left( \text{Slope} \right) = 1 \cdot e^{-3x} - 3xe^{-3x} = 0
\]

\[
e^{-3x} (1 - 3x) = 0
\]

\[
x = \frac{1}{3}.
\]

For this \(x\)-value,

\[
\text{Slope} = \frac{1}{3} e^{-3(1/3)} = \frac{1}{3} e^{-1} = \frac{1}{3e}.
\]

The formula for the slope shows that the slope tends toward 0 as \(x \to 0\). Thus the only critical point, \(x = 1/3\), must give a local and global maximum.

34. If 1200 cm of material is available to make a box with a square base and an open top, nd the largest possible volume of the box.

Solution:

Let \(b\) be the length of the base of the box and \(h\) the height. The surface area is \(1200 = b^2 + 4bh\) \(\Rightarrow\) \(h = (1200 - b^2)/(4b)\).

The volume is \(V = b^2h = b^2(1200 - b^2)/4b = 300b - b^3/4 \Rightarrow V'(b) = 300 - 3b^2\).

\(V'(b) = 0 \Rightarrow 300 = \frac{3}{4}b^2 \Rightarrow b^2 = 400 \Rightarrow b = \sqrt{400} = 20\). Since \(V'(b) > 0\) for \(0 < b < 20\) and \(V'(b) < 0\) for \(b > 20\), there is an absolute maximum when \(b = 20\) by the First Derivative Test for Absolute Extreme Values (see page 225).

If \(b = 20\), then \(h = (1200 - 20^2)/(4 \cdot 20) = 10\), so the largest possible volume is \(b^2h = (20)^2(10) = 4000\) cm\(^3\).
35. Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola \( y = 8 - x^2 \).

Solution:

The rectangle has area \( A(x) = 2xy = 2x(8 - x^2) = 16x - 2x^3 \), where

\[ 0 \leq x \leq 2\sqrt{2} \].

Now \( A'(x) = 16 - 6x^2 = 0 \) \( \Rightarrow x = 2\sqrt{2} \). Since

\( A(0) = A(2\sqrt{2}) = 0 \), there is a maximum when \( x = 2\sqrt{2} \). Then \( y = \frac{40}{9} \), so the rectangle has dimensions \( 4\sqrt{2} \) and \( \frac{10}{3} \).

36. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) A minimum?

Solution:

Let \( x \) be the length of the wire used for the square. The total area is

\[
A(x) = \left(\frac{x}{4}\right)^2 + \frac{1}{2} \left(\frac{10 - x}{3}\right) \sqrt{3} \left(\frac{10 - x}{3}\right)
\]

\[= \frac{1}{16}x^2 + \frac{\sqrt{3}}{6} (10 - x)^2 \quad 0 \leq x \leq 10
\]

\( A'(x) = \frac{1}{8}x - \frac{\sqrt{3}}{18} (10 - x) = 0 \) \( \iff \frac{\sqrt{3}}{9}x + \frac{4\sqrt{3}}{72}x - \frac{40\sqrt{3}}{72} = 0 \) \( \iff x = \frac{40\sqrt{3}}{9 + 4\sqrt{3}} \)

Now \( A(0) = \left(\frac{\sqrt{3}}{9}\right)100 \approx 4.81 \), \( A(10) = \frac{100}{16} = 6.25 \) and \( A(\frac{40\sqrt{3}}{9 + 4\sqrt{3}}) \approx 2.72 \), so

(a) The maximum area occurs when \( x = 10 \) m, and all the wire is used for the square.

(b) The minimum area occurs when \( x = \frac{40\sqrt{3}}{9 + 4\sqrt{3}} \approx 4.35 \) m.

37. Find \( f \) if \( f''(\theta) = \sin \theta + \cos \theta \) with \( f(0) = 3 \) and \( f'(0) = 4 \).

Solution:

\[
f''(\theta) = \sin \theta + \cos \theta \quad \Rightarrow \quad f'(\theta) = -\cos \theta + \sin \theta + C.
\]

\( f'(0) = -1 + C \) and \( f'(0) = 4 \) \( \Rightarrow C = 5 \), so

\[
f'(\theta) = -\cos \theta + \sin \theta + 5 \quad \text{and hence}, \quad f'(\theta) = -\sin \theta - \cos \theta + 5D.
\]

\( f(0) = -1 + D \) and \( f(0) = 3 \) \( \Rightarrow D = 4 \), so

\[
f(\theta) = -\sin \theta - \cos \theta + 5D + 4.
\]

38. A particle is moving with the acceleration \( a(t) = 10 + 3t - 3t^2 \). If \( s(0) = 0 \) and \( s(2) = 10 \) find \( s(t) \).

Solution:

\[
a(t) = v'(t) = 10 + 3t - 3t^2 \quad \Rightarrow \quad v(t) = 10t + \frac{3}{2}t^2 - \frac{3}{4}t^3 + C.
\]

\( s(t) = 5t^2 + \frac{1}{2}t^3 - \frac{1}{4}t^4 + Ct + D \)

\[0 = s(0) = D \quad \text{and} \quad 10 = s(2) = 20 + 4 - 4 + 2C \quad \Rightarrow \quad C = -5 \), so \( s(t) = -5t + 5t^2 + \frac{1}{2}t^3 - \frac{1}{4}t^4 \).

39. Figure below gives your velocity during a trip starting from home. Positive velocities take you away from home and negative velocities take you toward home. Where are you at the end of the 5 hours? When are you farthest from home? How far away are you at that time?
Solution: From \( t = 0 \) to \( t = 3 \), you are moving away from home \((v > 0)\); thereafter you move back toward home. So you are the farthest from home at \( t = 3 \). To find how far you are then, we can measure the area under the \( v \) curve as about 9 squares, or \( 9 \cdot 10 \) km/hr \( \cdot \) hr = 90 km. To find how far away from home you are at \( t = 5 \), we measure the area from \( t = 3 \) to \( t = 5 \) as about 25 km, except that this distance is directed toward home, giving a total distance from home during the trip of \( 90 - 25 = 65 \) km.

40. Figure below shows a Riemann sum approximation with \( n \) subdivisions to \( \int_a^b f(x) \, dx \).

(a) Is it a left- or right-hand approximation? Would the other one be larger or smaller?
(b) What are \( a \), \( b \), \( n \) and \( \Delta x \)?

![Riemann sum diagram]

Solution:

(a) Left-hand sum. Right-hand sum would be smaller.
(b) We have \( a = 0 \), \( b = 2 \), \( n = 6 \), \( \Delta x = \frac{2}{6} = \frac{1}{3} \).

41. Write out the terms of the right-hand sum with \( n = 5 \) that could be used to approximate \( \int_3^7 \frac{1}{1+x} \, dx \). Do not evaluate the terms or the sum.Is this an over or underestimate?

Solution: Since we have 5 subdivisions,

\[ \Delta x = \frac{b - a}{n} = \frac{7 - 3}{5} = 0.8. \]

The interval begins at \( x = 3 \) and ends at \( x = 7 \). Table gives the value of \( f(x) \) at the pertinent points.

<table>
<thead>
<tr>
<th>( x )</th>
<th>3.0</th>
<th>3.8</th>
<th>4.6</th>
<th>5.4</th>
<th>6.2</th>
<th>7.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>( \frac{1}{1+3.0} )</td>
<td>( \frac{1}{1+3.8} )</td>
<td>( \frac{1}{1+4.6} )</td>
<td>( \frac{1}{1+5.4} )</td>
<td>( \frac{1}{1+6.2} )</td>
<td>( \frac{1}{1+7.0} )</td>
</tr>
</tbody>
</table>

So a right-hand sum is

\[
\frac{1}{1+3.0}(0.8) + \frac{1}{1+3.8}(0.8) + \cdots + \frac{1}{1+7.0}(0.8).
\]

42. Given \( \int_{-2}^0 f(x) \, dx = 4 \) and Figure below, estimate:

(a) \( \int_0^2 f(x) \, dx \)
(b) \( \int_{-2}^2 f(x) \, dx \)
(c) The total shaded area.
Solution: The region shaded between \( x = 0 \) and \( x = 2 \) appears to have approximately the same area as the region shaded between \( x = -2 \) and \( x = 0 \), but it lies below the axis. Since \( \int_{-2}^{0} f(x)dx = 4 \), we have the following results:

(a) \( \int_{0}^{2} f(x)dx \approx -\int_{-2}^{0} f(x)dx = -4. \)

(b) \( \int_{-2}^{2} f(x)dx \approx 4 - 4 = 0. \)

(c) The total area shaded is approximately \( 4 + 4 = 8. \)

43. Use the Fundamental Theorem of Calculus to compute the following definite integrals.

(a) \( \int_{1}^{3} 2tdt \)

Solution: An antiderivative for \( 2t \) is \( F(t) = t^2 \), so by the Fundamental Theorem of Calculus,

\[
\int_{1}^{3} 2t \, dt = F(3) - F(1) = 9 - 1 = 8.
\]

(b) \( \int_{2}^{5} 6t + 4dt \)

Solution:

We have \( F(t) = 3t^2 + 4t \), so by the Fundamental Theorem of Calculus,

\[
\int_{2}^{5} (6t + 4) \, dt = F(5) - F(2) = 95 - 20 = 75.
\]

(c) \( \int_{1}^{5} \frac{1}{t} dt \)

Solution: We have \( F(t) = \ln t \), so by the Fundamental Theorem of Calculus,

\[
\int_{1}^{5} \frac{1}{t} \, dt = F(5) - F(1) = \ln 5 - \ln 1 = \ln 5.
\]

(d) \( \int_{0}^{\pi/2} \cos t \, dt \)

Solution: We have \( F(t) = \sin t \), so by the Fundamental Theorem of Calculus,

\[
\int_{0}^{\pi/2} \cos t \, dt = F(\pi/2) - F(0) = 1 - 0 = 1.
\]

(e) \( \int_{2}^{3} 7\ln(4) \cdot 4^t \, dt \)

Solution: We have \( F(t) = \frac{7}{4} \cdot 4^t \), so by the Fundamental Theorem of Calculus,

\[
\int_{2}^{3} 7\ln(4) \cdot 4^t \, dt = F(3) - F(2) = 448 - 112 = 336.
\]

(f) \( \int_{0}^{2} (3x^2 + 1) \, dx \).

Solution: By the Fundamental Theorem of Calculus since \( F(x) = x^3 + x \), we have

\[
\int_{0}^{2} (3x^2 + 1) \, dx = F(2) - F(0) = (2^3 + 2) - (0^3 + 0) = 10.
\]
44. The graphs in Figure ?? represent the velocity, \( v \), of a particle moving along the \( x \)-axis for time \( 0 \leq t \leq 5 \). The vertical scales of all graphs are the same. Identify the graph showing which particle:

(a) Has a constant acceleration.
(b) Ends up farthest to the left of where it started.
(c) Ends up the farthest from its starting point.
(d) Experiences the greatest initial acceleration.
(e) Has the greatest average velocity.
(f) Has the greatest average acceleration.

\[ \begin{align*}
\text{Graph (i)} & \quad \text{Graph (ii)} \\
\text{Graph (iii)} & \quad \text{Graph (iv)} \\
\text{Graph (v)} &
\end{align*} \]

(a) \( V \), since the slope is constant.
(b) \( IV \), since the net area under this curve is the most negative.
(c) \( III \), since the area under the curve is largest.
(d) \( II \), since the steepest ascent at \( t = 0 \) occurs on this curve.
(e) \( III \), since average velocity is \((\text{total distance})/5\), and \( III \) moves the largest total distance.

(f) \( I \), since average acceleration is \( \frac{1}{5} \int_0^5 v'(t) \, dt = \frac{1}{5} (v(5) - v(0)) \), and in \( I \), the velocity increases the most from start \( (t = 0) \) to finish \( (t = 5) \).

The graph of a continuous function \( f \) is given in Figure. Rank the following integrals in ascending numerical order. Explain your reasons.

45. (a) \( \int_0^2 f(x) \, dx \)
(b) \( \int_0^1 f(x) \, dx \)
(c) \( \int_0^2 (f(x))^{1/2} \, dx \)
(d) \( \int_0^2 (f(x))^2 \, dx \).
Solution: All the integrals have positive values, since $f \geq 0$. The integral in (ii) is about one-half the integral in (i), due to the apparent symmetry of $f$. The integral in (iv) will be much larger than the integral in (i), since the two peaks of $f^2$ rise to 10,000. The integral in (iii) will be smaller than half of the integral in (i), since the peaks in $f^{1/2}$ will only rise to 10. So

$$
\int_0^2 (f(x))^{1/2} \, dx < \int_0^1 f(x) \, dx < \int_0^2 f(x) \, dx < \int_0^2 (f(x))^2 \, dx.
$$

46. The graph of $g$ consists of two straight lines and a semicircle. Use it to evaluate each integral.

(a) $\int_0^2 g(x) \, dx$  (b) $\int_2^6 g(x) \, dx$  (c) $\int_0^7 g(x) \, dx$

Solution:

(a) $\int_0^2 g(x) \, dx = \frac{1}{2} \cdot 4 \cdot 2 = 4$ (area of a triangle)

(b) $\int_2^6 g(x) \, dx = -\frac{1}{2} \pi (2)^2 = -2\pi$ (negative of the area of a semicircle)

(c) $\int_0^7 g(x) \, dx = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$ (area of a triangle)

$$\int_0^7 g(x) \, dx = \int_0^2 g(x) \, dx + \int_2^6 g(x) \, dx + \int_6^7 g(x) \, dx = 4 - 2\pi + \frac{1}{2} = 4.5 - 2\pi$$

47. Evaluate the integral by interpreting in terms of areas

$$\int_0^{10} |x - 5| \, dx$$

Solution:
\[ \int_{0}^{10} \left| x - 5 \right| \, dx \] can be interpreted as the sum of the areas of the two shaded triangles; that is, \(2 \left( \frac{1}{2} \right) (5) = 25.\)

48. Evaluate the following integrals:

1. \[ \int_{-1}^{3} x^5 \, dx \]
2. \[ \int_{-1}^{3} (1 + 2x - 4x^3) \, dx \]
3. \[ \int_{0}^{2} (6x^2 - 4x + 5) \, dx \]
4. \[ \int_{-2}^{0} (u^5 - u^3 + u^2) \, du \]
5. \[ \int_{0}^{1} x^{4/3} \, dx \]
6. \[ \int_{1}^{0} \sqrt{x} \, dx \]
7. \[ \int_{-1}^{0} (2x - e^x) \, dx \]
8. \[ \int_{\pi}^{2\pi} \cos \theta \, d\theta \]
9. \[ \int_{-2}^{2} (3u + 1)^2 \, du \]
10. \[ \int_{0}^{4} (2v + 5)(3v - 1) \, dv \]
11. \[ \int_{-2}^{-1} \left( 4y^3 + \frac{2}{y^3} \right) \, dy \]
12. \[ \int_{1}^{2} \frac{y + 5y^7}{y^3} \, dy \]
49. Use part I of Fundamental Theorem of Calculus to find the derivative of the following functions:

(a) \( g(x) = \int_0^x \sqrt{1 + 2t} \, dt \)

(b) \( g(x) = \int_1^x \ln t \, dt \)

(c) \( g(y) = \int_2^y t^2 \sin t \, dt \)

Solution:

\[ f(t) = \sqrt{1 + 2t} \] and \( g(x) = \int_0^x \sqrt{1 + 2t} \, dt \), so by FTC1, \( g'(x) = f(x) = \sqrt{1 + 2x} \).

\( f(t) = \ln t \) and \( g(x) = \int_1^x \ln t \, dt \), so by FTC1, \( g'(x) = f(x) = \ln x \).

\( f(t) = t^2 \sin t \) and \( g(y) = \int_2^y t^2 \sin t \, dt \), so by FTC1, \( g'(y) = f(y) = y^2 \sin y \).

50. Find the interval on which the curve

\[ y = \int_0^x \frac{1}{1 + t + t^2} \, dt \]
51. If $f(1) = 12$, $f'$ is continuous and $\int_1^4 f'(x) \, dx = 17$ what is $f(4)$?

Solution:

By FTC2, $\int_1^4 f'(x) \, dx = f(4) - f(1)$, so $17 = f(4) - 12 \Rightarrow f(4) = 17 + 12 = 29.$

52. Evaluate the following indefinite integrals using substitution method.

7. $\int 2x(x^2 + 3)^4 \, dx$  
8. $\int x^2(x^3 + 5)^9 \, dx$

9. $\int (3x - 2)^{20} \, dx$  
10. $\int xe^{x^2} \, dx$

11. $\int \frac{(\ln x)^2}{x} \, dx$  
12. $\int (2 - x)^6 \, dx$

Solution:

7. Let $u = x^2 + 3$. Then $du = 2x \, dx$, so $\int 2x(x^2 + 3)^4 \, dx = \int u^4 \, du = \frac{1}{5} u^5 + C = \frac{1}{5} (x^2 + 3)^5 + C.$

8. Let $u = x^3 + 5$. Then $du = 3x^2 \, dx$ and $x^2 \, dx = \frac{1}{3} \, du$, so $\int x^2(x^3 + 5)^9 \, dx = \int u^9 \left(\frac{1}{3} \, du\right) = \frac{1}{3} \cdot \frac{1}{10} u^{10} + C = \frac{1}{30} (x^3 + 5)^{10} + C.$

9. Let $u = 3x - 2$. Then $du = 3 \, dx$ and $dx = \frac{1}{3} \, du$, so $\int (3x - 2)^{20} \, dx = \int u^{20} \left(\frac{1}{3} \, du\right) = \frac{1}{3} \cdot \frac{1}{21} u^{21} + C = \frac{1}{63} (3x - 2)^{21} + C.$

10. Let $u = x^2$. Then $du = 2x \, dx$, so $\int xe^{x^2} \, dx = \int e^{u} \left(\frac{1}{2} \, du\right) = \frac{1}{2} e^u + C = \frac{1}{2} e^{x^2} + C.$

11. Let $u = \ln x$. Then $du = \frac{dx}{x}$, so $\int \frac{(\ln x)^2}{x} \, dx = \int u^2 \, du = \frac{1}{3} u^3 + C = \frac{1}{3} (\ln x)^3 + C.$

12. Let $u = 2 - x$. Then $du = -dx$ and $dx = -du$, so $\int (2 - x)^6 \, dx = \int u^6 (-du) = -\frac{1}{7} u^7 + C = -\frac{1}{7} (2 - x)^7 + C.$
53. Evaluate the following definite integral using substitution method.

35. \[ \int_0^2 (x - 1)^{25} \, dx \]

36. \[ \int_0^7 \sqrt{4 + 3x} \, dx \]

37. \[ \int_0^1 x^2(1 + 2x^3)^5 \, dx \]

38. \[ \int_0^{\sqrt{\pi}} x \cos(x^2) \, dx \]

Solution:

35. Let \( u = x - 1 \), so \( du = dx \). When \( x = 0 \), \( u = -1 \); when \( x = 2 \), \( u = 1 \). Thus, \( \int_0^2 (x - 1)^{25} \, dx = \int_{-1}^{1} u^{25} \, du = 0 \) by Theorem 7(b), since \( f(u) = u^{25} \) is an odd function.

36. Let \( u = 4 + 3x \), so \( du = 3 \, dx \). When \( x = 0 \), \( u = 4 \); when \( x = 7 \), \( u = 25 \). Thus,

\[ \int_0^7 \sqrt{4 + 3x} \, dx = \int_4^{25} \sqrt{u} \left( \frac{1}{3} \, du \right) = \frac{1}{3} \left[ \frac{u^{3/2}}{3/2} \right]_4^{25} = \frac{2}{9} \left( 25^{3/2} - 4^{3/2} \right) = \frac{2}{9} \left( 25^3 - 4^3 \right) = \frac{254}{9} = 26. \]

37. Let \( u = 1 + 2x^3 \), so \( du = 6x^2 \, dx \). When \( x = 0 \), \( u = 1 \); when \( x = 1 \), \( u = 3 \). Thus,

\[ \int_0^1 x^2(1 + 2x^3)^5 \, dx = \int_1^3 u^5 \left( \frac{1}{6} \, du \right) = \frac{1}{6} \left[ \frac{1}{5} u^6 \right]_1^3 = \frac{1}{30} \left( 3^6 - 1^6 \right) = \frac{1}{30} \left( 729 - 1 \right) = \frac{728}{30} = \frac{182}{9}. \]

38. Let \( u = x^2 \), so \( du = 2x \, dx \). When \( x = 0 \), \( u = 0 \); when \( x = \sqrt{\pi} \), \( u = \pi \). Thus,

\[ \int_0^{\sqrt{\pi}} x \cos(x^2) \, dx = \int_0^{\pi} \cos u \left( \frac{1}{2} \, du \right) = \frac{1}{2} \left[ \sin u \right]_0^\pi = \frac{1}{2} \left( \sin \pi - \sin 0 \right) = \frac{1}{2} (0 - 0) = 0. \]