Solutions to Final Exam

Math 1a
Introduction to Calculus

May 23, 2008
1. (10 Points) Answer the questions as completely as possible. For instance, if a limit does not exist but the left- and right-hand limits do, say so. If a limit does not exist but is infinite, say so. (Each part is worth two points; one for the answer and one for the reasoning.)

(i) \( \lim_{x \to \infty} \frac{x^4 + 1}{x^3 - 1} \)

**Solution.** The degree of the numerator is bigger than the degree of the denominator, so the limit is \( \infty \). Alternatively, one could use L'Hôpital's Rule (four times). ▲

(ii) \( \lim_{x \to -\infty} \frac{e^x}{e^x + 1} \)

**Solution.** As \( x \to -\infty \), the numerator tends to zero while the denominator tends to \( 0 + 1 = 1 \). So the quotient tends to zero. ▲

(iii) \( \lim_{x \to 0} \frac{\sin 3x}{2x} \)

**Solution.** We can change the variable to \( y = 3x \). Then

\[
\lim_{x \to 0} \frac{\sin 3x}{2x} = \lim_{x \to 0} \frac{3}{2} \cdot \lim_{x \to 0} \frac{\sin 3x}{3x} = \frac{3}{2} \cdot 1 = \frac{3}{2}.
\]

▲

(iv) \( \lim_{x \to 1} \frac{\ln x}{x - 1} \)

**Solution.** Since the numerator and denominator both tend to 1, L'Hôpital's Rule is appropriate here:

\[
\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{1/x}{1} = 1.
\]

Actually, this is equivalent to the fact that the derivative of the natural logarithm at \( x = 1 \) is 1. ▲

(v) \( \lim_{x \to \pi/4} \frac{2 \sin x + \sqrt{2}}{2 \sin x - \sqrt{2}} \)

**Solution.** As \( x \to \pi/4 \), the denominator tends to 0 but the numerator tends to \( 2\sqrt{2} \). So the limit is probably some kind of infinity. To see which, take the one-sided limits. As \( x \to \pi/4^- \), \( 2 \sin x - \sqrt{2} \) is negative but close to zero. So the limit from this side is \( -\infty \). Likewise, the limit from the positive side is \( +\infty \). ▲
2. (10 Points) Consider the function

\[ f(x) = \begin{cases} 
2x + 1 & x \leq -1 \\
2 - 1 & -1 < x \leq 1 \\
2x - 2 & 1 < x \leq 2 \\
-x^2 + 4x - 2 & x > 2 
\end{cases} \]

(a) For which \( x \) is \( f \) continuous?

\textit{Solution.} By the direct substitution property, \( f(x) \) is continuous at all \( x \) “inside” the intervals of definition. The only points left to consider are the crossover points \(-1, 1, \) and \(2.

Still by the direct substitution property, we have

\[
\begin{align*}
\lim_{x \to -1^-} f(x) &= \lim_{x \to -1^-} 2x + 1 = -1 \\
\lim_{x \to -1^+} f(x) &= \lim_{x \to -1^+} (2x - 1) = 0 \\
\lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} 2x - 2 = 2 \\
\lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} -x^2 + 4x - 2 = 2
\end{align*}
\]

So \( f \) is also continuous at \(1\) and \(2\). In summary, \( f \) is continuous at all real numbers except \(-1\). ▲

(b) For which \( x \) is \( f \) differentiable?

\textit{Solution.} Again by the direct substitution property, \( f \) is automatically differentiable at all points in the intervals \((-\infty, -1), (-1, 1), (1, 2), \) and \((2, \infty)\). Since \( f \) is not continuous at \(-1\), it cannot be differentiable there. We need only check \(1\) and \(2\). We have

\[
\begin{align*}
\lim_{x \to -1^-} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \to -1^-} \frac{(2x - 1) - 0}{x - 1} = \lim_{x \to -1^-} (x + 1) = 2 \\
\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \to 1^-} \frac{(2x - 2) - 0}{x - 1} = \lim_{x \to 1^-} 2 = 2 \\
\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} &= \lim_{x \to 2^-} \frac{(2x - 2) - 2}{x - 2} = \lim_{x \to 2^-} 2 = 2 \\
\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} &= \lim_{x \to 2^+} \frac{-x^2 + 4x - 2 - 2}{x - 2} = \lim_{x \to 2^+} -(x - 2) = 0
\end{align*}
\]

So \( f'(1) = 2 \) and \( f \) is not differentiable at \(2\). In summary, \( f \) is differentiable at all real numbers except \(-1\) and \(2\). ▲

Here is a plot of \( f(x) \):
Remarks. It is acceptable to argue that since \( \lim_{x \to 1^+} f'(x) = 2 \) and \( \lim_{x \to 1^-} f'(x) = 2 \), then \( f \) is differentiable at 1 and \( f'(1) = 2 \). This isn’t exactly from the definition but you can use the Mean Value Theorem to show it.
3. (10 Points) Let \( f(x) = \frac{6}{x^2 + 1} \). Use the definition of the derivative to find \( f'(2) \). You may not use any derivative shortcuts on this problem. (They are allowed on every other problem, though.)

Solution. There are two limit expressions for the derivative. One is

\[
\lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0} \frac{\frac{6}{(2+h)^2+1} - \frac{6}{5}}{h} = \lim_{h \to 0} \frac{30 - 6[(2+h)^2+1]}{5h[(2+h)^2+1]}
\]

\[
= \lim_{h \to 0} \frac{30 - 6(h^2 + 4h + 5)}{5h(h^2 + 4h + 5)} = \lim_{h \to 0} \frac{-6(h + 4)}{5h(h^2 + 4h + 5)}
\]

\[
= -\frac{6 \cdot 4}{5 \cdot 5} = -\frac{24}{25}.
\]

Another one is

\[
\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{\frac{6}{x^2 + 1} - \frac{6}{5}}{x - 2} = \lim_{x \to 2} \frac{30 - 6(x^2 + 1)}{5(x^2 + 1)(x - 2)}
\]

\[
= \lim_{x \to 2} \frac{24 - 6x^2}{5(x^2 + 1)(x - 2)} = \lim_{x \to 2} \frac{6(2-x)(x+1)}{5(x^2 + 1)(x - 2)}
\]

\[
= -\frac{6 \cdot 4}{5 \cdot 5} = -\frac{24}{25}.
\]
4. (10 Points) Find derivatives of the following functions of $x$.

(i) $3x^2 + 1$

Solution. By the power rule, the derivative is $6x$. ▲

(ii) $\frac{x}{3x^2 + 1}$

Solution. By the quotient rule, the derivative is

$$\frac{(3x^2 + 1)(1) - x(6x)}{(3x^2 + 1)^2} = \frac{-3x^2 + 1}{(3x^2 + 1)^2}$$

▲

(iii) $\sec^3 x$

Solution. By the chain rule, the derivative is

$$3(\sec^2 x)(\sec x \tan x) = 3 \sec^3 x \tan x$$

▲

(iv) $2^{\sin^2 x + \cos^2 x - 1}$

Solution. By the pythagorean theorem, the exponent is $1 - 1 = 0$. So the function is constant, meaning its derivative is zero. ▲

(v) $e^{x^2 \sin x}$

Solution. We have

$$e^{x^2 \sin x} \left(2x \sin x + x^2 \cos x\right)$$

▲
5. (10 Points) Consider the curve with equation

\[ y^2 = x^2(x + 1) \]

(i) (4 points) Find the slope of the line tangent to the curve through the point \((3, -6)\).

Solution. Implicitly differentiating the relation, we have

\[
2y \frac{dy}{dx} = 3x^2 + 2x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{3x^2 + 2x}{2y}
\]

At \((3, -6)\), this expression evaluates to \(-\frac{33}{12} = \frac{11}{4}\).

(ii) (3 points) Find all points on the curve to which the tangent line is horizontal.

Solution. The tangent line is horizontal when \(\frac{dy}{dx} = 0\). The expression \(\frac{dy}{dx} = \frac{x(3x + 2)}{2y}\) is zero when the numerator is zero and the denominator is not zero. Since when \(x = 0\), \(y = 0\), the only solution is \(x = -2/3\). Hence the derivative is zero at the two points \((2/3, 2\sqrt{3})\).

(iii) (3 points) Find all points on the curve to which the tangent line is vertical.

Solution. The tangent line is vertical when \(\frac{dx}{dy} = 0\). The expression \(\frac{dx}{dy} = \frac{2y}{x(3x + 2)}\) is zero when the numerator is zero and the denominator is not zero. The only solution is \(y = 0\) and \(x = -1\).
6. (10 Points) Match the following functions to their graphs. (One of the graphs is not used.)

(i) \( f(x) = x^4 - 2x^2 + 1 \) \[ \text{D} \]

(ii) \( f(x) = \frac{3x^4}{2} + \frac{x^3}{4} - 3x^2 - \frac{3x}{4} + 1 \) \[ \text{E} \]

(iii) \( f(x) = -\frac{3x^4}{2} - \frac{x^3}{4} + 3x^2 + \frac{3x}{4} - 1 \) \[ \text{A} \]

(iv) \( f(x) = \frac{3x^4}{2} - \frac{x^3}{4} - 3x^2 + \frac{3x}{4} + 1 \) \[ \text{B} \]

(v) \( f(x) = \frac{3x^4}{16} + \frac{x^3}{4} - \frac{3x^2}{8} - \frac{3x}{4} - \frac{5}{16} \) \[ \text{F} \]
7. (10 Points) A fence $h$ ft high runs parallel to a tall building and $w$ ft from it. We will find the length of the shortest ladder that will reach from the ground across the top of the fence to the wall of the building.

(a) (4 points) Find the length of the ladder as a function of $\theta$. Call this function $\ell(\theta)$. What is the domain of $\ell$?

Solution. Let $\ell_1$ and $\ell_2$ be as shown, so $\ell = \ell_1 + \ell_2$. We have

\[
\sin \theta = \frac{h}{\ell_1} \implies \ell_1 = h \csc \theta \tag{1}
\]
\[
\cos \theta = \frac{w}{\ell_2} \implies \ell_2 = w \sec \theta \tag{2}
\]

The domain is $0 < \theta < \frac{\pi}{2}$.

(b) (4 points) Find the point at which $\ell$ is minimized. Make sure you show it is a minimum!

(Your answer will involve $w$ and $h$.)

Solution. To find the critical points of $\ell$, we solve $\ell'(\theta) = 0$:

\[
0 = -h \csc \theta \cot \theta + w \sec \theta \tan \theta
\]
\[
h \csc \theta \cot \theta = w \sec \theta \tan \theta
\]
\[
\frac{h \cos \theta}{\sin^2 \theta} = w \frac{\sin \theta}{\cos^2 \theta}
\]
\[
\frac{h}{w} = \frac{\sin^3 \theta}{\cos^3 \theta} \implies \theta = \arctan \left( \frac{h}{w} \right)^{1/3}
\]

It turns out that

\[
\ell'' = h \left( \csc^3(\theta) + \cot^2(\theta) \csc(\theta) \right) + w \left( \sec^3(\theta) + \tan^2(\theta) \sec(\theta) \right)
\]

and this is positive for all $\theta$ with $0 < \theta < \frac{\pi}{2}$. So the critical point is the global minimum.

(c) (2 points) Find the minimum value of $\ell$. 

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Solution. If \( \tan \theta = \frac{h^{1/3}}{w^{1/3}} \), it follows from trig identities (or you can draw a right triangle) that

\[
\csc \theta = \frac{\sqrt{h^{2/3} + w^{2/3}}}{h^{1/3}} \quad \text{sec} \theta = \frac{\sqrt{h^{2/3} + w^{2/3}}}{w^{1/3}}
\]

So

\[
\ell(\theta) = h \cdot \frac{\sqrt{h^{2/3} + w^{2/3}}}{h^{1/3}} + w \cdot \frac{\sqrt{h^{2/3} + w^{2/3}}}{w^{1/3}} = h^{2/3} \sqrt{h^{2/3} + w^{2/3}} + w^{2/3} \sqrt{h^{2/3} + w^{2/3}}
\]

\[
= \left( h^{2/3} + w^{2/3} \right) \sqrt{h^{2/3} + w^{2/3}} = \left( h^{2/3} + w^{2/3} \right)^{3/2}
\]
8. (10 Points) Consider the definite integral

\[ I = \int_{0}^{1} \left( \frac{x^4}{2} + \frac{8x^3}{5} \right) \, dx \]

In these parts, you may leave your answer as a fraction or sum of fractions.

(i) (2 points) Find \( L_2 \), the approximation to \( I \) using two subintervals and left endpoints in the Riemann sum.

Solution. We have

\[
L_2 = \frac{1}{2} \left( \frac{0^4}{2} + \frac{8 \cdot 0^3}{5} \right) + \frac{1}{2} \left( \frac{(1/2)^4}{2} + \frac{8(1/2)^3}{5} \right) = \frac{37}{320}
\]

(ii) (2 points) Find \( M_2 \), the approximation to \( I \) using midpoints in the Riemann sum.

Solution. We have

\[
L_2 = \frac{1}{2} \left( \frac{(1/4)^4}{2} + \frac{8(1/4)^3}{5} \right) + \frac{1}{2} \left( \frac{(3/4)^4}{2} + \frac{8(3/4)^3}{5} \right) = \frac{1101}{2560}
\]

(iii) (3 points) Find \( S_2 \), the approximation to \( I \) using two divisions and Simpson’s Rule.

Solution. We have

\[
S_2 = \frac{1}{6} \left[ \left( \frac{0^4}{2} + \frac{8 \cdot 0^3}{5} \right) + 4 \left( \frac{(1/2)^4}{2} + \frac{8(1/2)^3}{5} \right) + \left( \frac{1^4}{2} + \frac{8 \cdot 1^3}{5} \right) \right] = \frac{121}{240}
\]

(iv) (3 points) Find \( I \) exactly using the Fundamental Theorem of Calculus.

Solution. We have

\[
I = \left[ \frac{x^5}{10} + \frac{2x^4}{5} \right]_0^1 = \frac{1}{10} + \frac{2}{5} = \frac{1}{2}
\]
9. (10 Points) Suppose that the power exerted by a participant in an indoor cycling class is given by the following graph:

![Graph of power exerted over time](image)

The total work performed by the cyclist as a function of $t$, measured from the beginning of the class, is

$$W(t) = \int_0^t P(s) \, ds$$

(i) Place the following numbers in order: $W(2)$, $W(4)$, and $W(7)$.

Solution. By the Fundamental Theorem of Calculus, $W'(t) = P(t)$. Since $P(t) > 0$ for all $t$, $W(t)$ is strictly increasing. This means $W(2) < W(4) < W(7)$.

(ii) If 2 units of work is done over the warmup interval, 4 units of work over the shift interval, and 18 units of work over the whole class, how much work is done over each speed burst? That is, what is $\int_4^5 P(s) \, ds$?

Solution. The three “speed burst” intervals are the same, so they each take up one unit of time and the same amount of work. If this amount is $w$, we have $2 + 4 + 3w = 18$, so $w = 4$.

(iii) At which points is the work changing most rapidly?

Solution. “Changing most rapidly” means that $W'(t)$ is maximized. These points are the peaks of the graph of $P(t) = W'(t)$ Using the regularity of the graph over the interval $[4, 7]$, we can see that they occur at $4.25, 5.25,$ and $6.25$.

(iv) Along which intervals is $W''(t) > 0$?

Solution. $W''(t) > 0$ where $P'(t) > 0$, i.e., where $P$ is increasing. This is apparently over the intervals $[2, 4], [4, 4.25], [4.75, 5.25], [5.75, 6.25], [6.75, 7]$.
(v) Which of these could be the graph of $W(t)$?

Answer: F

Solution. Only the graph (F) is always increasing.

▲
10. (10 Points)

(I) (5 points) Find the area between the curve \( y = x \cos(x^2) \) and the x-axis between \( x = 0 \) and \( x = \sqrt{\pi/2} \).

Solution. This area is computed by the integral \( \int_0^{\sqrt{\pi/2}} x \cos(x^2) \, dx \). To evaluate it, let \( u = x^2 \), so \( du = 2x \, dx \). Substitution yields

\[
\int_0^{\sqrt{\pi/2}} x \cos(x^2) \, dx = \frac{1}{2} \int_0^{\pi/2} \cos u \, du = \frac{1}{2} \sin u \bigg|_0^{\pi/2} = \frac{1}{2}. 
\]

▲

(II) Find the following integrals. In the case of a definite integral, your answer should be a real number. In the case of an indefinite integral, your answer should be the most general antiderivative.

(i) (2 points) \( \int \frac{e^u}{1 + e^u} \, du \)

Solution. The quickest substitution is \( y = 1 + e^u \). Then \( dy = e^u \, du \) so

\[
\int \frac{e^u}{1 + e^u} \, du = \int \frac{1}{y} \, dy = \ln |y| + C = \ln(1 + e^u) + C
\]

(The absolute value bars at the end aren’t necessary because the term in the logarithm is always positive.)

▲

(ii) (3 points) \( \int_1^2 \frac{\ln x}{x} \, dx \)

Solution. Let \( u = \ln x \), so \( du = \frac{1}{x} \, dx \). Then

\[
\int_1^2 \frac{\ln x}{x} \, dx = \int_{\ln 1}^{\ln 2} u \, du = \frac{u^2}{2} \bigg|_{\ln 1}^{\ln 2} = \frac{1}{2} (\ln 2)^2.
\]

Be careful: It’s \( \ln(2^2) \) which is equal to \( 2 \ln 2 \), not \( (\ln 2)^2 \).
11. (10 Points)

(I) (5 points) Suppose a bug is traveling along the x-axis. At time \( t = 0 \), the bug is one unit to the right of his home. His velocity at time \( t \) is \( v(t) = t \arctan t \), where positive means "to the right." How far is he from home at time \( t = 1 \)?

Solution. The number we want is \( s(1) = 1 + \int_0^1 t \arctan t \, dt \). To evaluate the integral, 
let \( u = \arctan t \) and \( dv = t \, dt \). Then \( du = \frac{dt}{1+t^2} \) and \( v = \frac{t^2}{2} \). So

\[
\int_0^1 t \arctan t \, dt = \frac{1}{2} t^2 \arctan t \bigg|_0^1 - \frac{1}{2} \int_0^1 \frac{t^2}{1+t^2} \, dt = \frac{1}{2} \cdot \frac{\pi}{4} - \frac{1}{2} \int_0^1 \frac{t^2}{1+t^2} \, dt
\]

To do the second integral, notice that \( \frac{t^2}{1+t^2} = 1 - \frac{1}{1+t^2} \). So

\[
\int_0^1 \frac{t^2}{1+t^2} \, dt = \int_0^1 \left( 1 - \frac{1}{1+t^2} \right) \, dt = (t - \arctan t) \bigg|_0^1 = 1 - \frac{\pi}{4}
\]

Putting this all together gives

\[
s(1) = 1 + \frac{\pi}{8} - \frac{1}{2} \left( 1 - \frac{\pi}{4} \right) = \frac{1}{2} + \frac{\pi}{4}
\]

▲

(II) Find the following integrals. In the case of a definite integral, your answer should be a real number. In the case of an indefinite integral, your answer should be the most general antiderivative.

(i) (3 points) \( \int_0^1 xe^x \, dx \)

Solution. Let \( u = x \), \( dv = e^x \, dx \). So \( du = dx \) and \( v = e^x \). Then

\[
\int_0^1 xe^x \, dx = xe^x \bigg|_0^1 - \int_0^1 e^x \, dx = (e^1 - 0) - (e^1 - e^0) = 1.
\]

▲

(ii) (2 points) \( \int \sqrt{x} \ln x \, dx \)

Solution. Let \( u = \ln x \), \( dv = \sqrt{x} \, dx \). Then \( du = \frac{dx}{x} \), \( v = \frac{2}{3} x^{3/2} \). So

\[
\int \sqrt{x} \ln x \, dx = \frac{2}{3} x^{3/2} \ln x - \frac{2}{3} \int x^{1/2} \, dx = \frac{2}{3} x^{3/2} \ln x - \frac{4}{9} x^{3/2} + C
\]

▲
12. (10 Points) Let $f$ be a continuous function. Show that for any $x,$

$$\int_0^x \left( \int_0^t f(s) \, ds \right) \, dt = \int_0^x f(t)(x-t) \, dt$$

*Hint.* Relax. Notice that both the left-hand and right-hand expressions are functions of $x.$ All the other variables are “dummy variables,” meaning they’re integrated over.

There is more than one way to show they are equal:

- Show that both sides have the same derivative and agree at a single point.
- Manipulate one side (integration by parts?) to get the other side.

**Solution.** Let $F(t) = \int_0^t f(s) \, ds,$ and let $G_1$ be the left-hand side of the equation above. We can now write this as $G_1(x) = \int_0^x F(t) \, dt.$ Let $G_2$ be the right-hand side of the equation. We want to show $G_1(x) = G_2(x)$ for all $x.$

To follow the first hint, notice $G_1'(x) = F(x).$ For the right-hand side, the integration is in the $t$ variable, but the $x$ appears *both* in the integrand and in the limit. To take its derivative, write it as

$$G_2(x) = x \int_0^x f(t) \, dt - \int_0^x tf(t) \, dt = xF(x) - \int_0^x tf(t) \, dt$$

Then

$$G_2'(x) = F(x) + xF'(x) - xf(x) = F(x)$$

So $G_1'(x) = G_2'(x).$ But also $G_1(0) = G_2(0) = 0.$ So the functions are equal.

For the other hint, integrate the right-hand side by parts. Let $u = (x-t)$ and $dv = f(t) \, dt.$ Then $du = -dt$ and $v = F(t).$ Then

$$\int_0^x f(x)(x-t) \, dt = (x-t)F(t)|_0^x + \int_0^x F(t) \, dt = 0 - xF(0) + \int_0^x F(t) \, dt$$

\[\square\]