Solutions to Final Examination

Math 1a
Introduction to Calculus

May 26, 2005

Show all of your work. Full credit may not be given for an answer alone. You may use the backs of the pages or the extra pages for scratch work. Do not unstaple or remove pages.

Problems are numbered with arabic numerals (1, 2, 3, \ldots) and may stretch across several pages. Parts of problems are enumerated with either letters ((a), (b), (c), \ldots) or small roman numerals ((i), (ii), (iii), \ldots).

Do not put your name on any page besides the first page. If you like, you may put your HUID on the top of each page you write on.

This is a non-calculator exam.

Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.

—Handbook for Students
1. (15 Points) Evaluate the following limits, if they exist. Indicate where you use the Direct Substitution Property or L’Hôpital’s Rule. If the limit does not exist, find the one-sided limits (if they exist).

(i) \( \lim_{x \to 0^+} \frac{\cos x}{x} \)

Solution. (3 points) The limit does not exist because it’s of the form \( \frac{1}{0} \). As \( x \) tends to zero from the positive side, the fraction looks like \( 1 \) divided by something small and positive. Hence

\[
\lim_{x \to 0^+} \frac{\cos x}{x} = \infty.
\]

Likewise,

\[
\lim_{x \to 0^-} \frac{\cos x}{x} = -\infty.
\]

(ii) \( \lim_{x \to 1} (x^2 - 4x + 1) \)

Solution. (2 points) By the Direct Substitution Property:

\[
\lim_{x \to 1} (x^2 - 4x + 1) = 1^2 - 4 \cdot 1 + 1 = -2.
\]

(iii) \( \lim_{x \to \infty} \left( x - \sqrt{x^2 + x} \right) \)

Solution. (5 points) The limit is of the form \( \infty - \infty \), so is indeterminate. We rationalize:

\[
\lim_{x \to \infty} \left( x - \sqrt{x^2 + x} \right) = \lim_{x \to \infty} \frac{x - \sqrt{x^2 + x} \left( x + \sqrt{x^2 + x} \right)}{\left( x + \sqrt{x^2 + x} \right)}
\]

\[
= \lim_{x \to \infty} \frac{x^2 - (x^2 + x)}{x + \sqrt{x^2 + x}}
\]

\[
= \lim_{x \to \infty} \frac{-x}{x + \sqrt{x^2 + x}}
\]

\[
= \lim_{x \to \infty} \frac{-1}{1 + \sqrt{1 + 1/x}} = -1/2.
\]

(iv) \( \lim_{x \to 0} \frac{\cos x - 1 + \frac{1}{2} x^2}{x^4} \)
Solution. (5 points) The limit is of the form $\frac{0}{0}$, so is indeterminate. We have

$$\lim_{x \to 0} \frac{\cos x - 1 + \frac{1}{2}x^2}{x^4} \text{ H } \lim_{x \to 0} \frac{-\sin x + x}{4x^3} \text{ H } \lim_{x \to 0} \frac{-\cos x + 1}{12x^2} \text{ H } \lim_{x \to 0} \frac{\sin x}{24x} \text{ H } \frac{1}{24} \lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{24}. $$
2. (20 Points) Find the following derivatives.

(i) \( \frac{d}{dx} (x^2 - x + 4) \)

Solution. (2 points) By the power rule:
\[
\frac{d}{dx} (x^2 - x + 4) = 2x - 1.
\]

(ii) \( \frac{d}{dx} (x^2 \sin x^2) \)

Solution. (4 points) By the product rule:
\[
\frac{d}{dx} (x^2 \sin x^2) = 2x \sin x^2 + x^2 \cos x^2 (2x) = 2x \sin x^2 + 2x^3 \cos x^2.
\]

(iii) \( \frac{d}{dx} \log_4 3^x \) (simplify your answer.)

Solution. (4 points) Using the chain rule, we have
\[
\frac{d}{dx} \log_4 3^x = \frac{1}{\ln 4} \frac{d}{dx} 3^x = \frac{1}{\ln 4} (3^x \ln 3)
\]
\[
= \frac{\ln 3}{\ln 4}.
\]
Shortcut: We have for any \( u > 0 \),
\[
\log_a u = \frac{\ln u}{\ln a},
\]
so
\[
\log_4 3^x = \frac{\ln 3^x}{\ln 4} = \frac{x \ln 3}{\ln 4} = \frac{\ln 3}{\ln 4} x.
\]
Thus
\[
\frac{d}{dx} \log_4 3^x = \frac{\ln 3}{\ln 4}.
\]

(iv) \( \frac{dy}{dx} \), where \( y^2 = \frac{x}{x+1} \) (your answer should involve \( x \) and \( y \).)
Solution. (5 points) Implicitly differentiating the relation between $x$ and $y$:

\[
\frac{dy}{dx} = \frac{(x + 1) \cdot 1 - x \cdot 1}{(x + 1)^2} = \frac{1}{(x + 1)^2} \quad \frac{dy}{dx} = \frac{1}{2y(x + 1)^2}.
\]

\(v\) \(\frac{dy}{dx}\), where \(y = (1 + 1/x)^x\) (your answer should not involve \(y\)).

Solution. (5 points) Taking the logarithm of the equation for \(y\) in terms of \(x\) gives:

\[
\ln y = x \ln \left(1 + \frac{1}{x}\right)
\]

so,

\[
\frac{1}{y} \cdot \frac{dy}{dx} = \ln \left(1 + \frac{1}{x}\right) + \frac{x}{1 + \frac{1}{x}} \left(-\frac{1}{x^2}\right)
= \ln \left(1 + \frac{1}{x}\right) - \frac{1}{x + 1}.
\]

Therefore,

\[
\frac{dy}{dx} = \left\{ \ln \left(1 + \frac{1}{x}\right) - \frac{1}{x + 1} \right\} \left(1 + \frac{1}{x}\right)^x
\]

\(\square\)
3. (10 Points) Find the equations of both lines through the origin which are tangent to the circle

\[(x - 2)^2 + y^2 = 1.\]

**Solution.** The equation describes a circle with center \((2, 0)\) and radius 1. If \((x, y)\) is on the circle, \(x\) and \(y\) satisfy the equation above. If additionally the line through \((0, 0)\) and \((x, y)\) is tangent to the circle, we have

\[
\frac{y}{x} = \frac{dy}{dx}\bigg|_{(x,y)}.
\]

By implicit differentiation we have

\[
2(x - 2) + 2y \frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx} = \frac{2 - x}{y}.
\]

Thus

\[
\frac{y}{x} = \frac{2 - x}{y} \implies y^2 = x(2 - x).
\]

We can solve both equations for \(y^2\):

\[
1 - (x - 2)^2 = x(2 - x)
\]

\[
-x^2 + 4x - 3 = -x^2 + 2x
\]

\[
2x - 3 = 0
\]

\[
x = \frac{3}{2}.
\]

The corresponding \(y\)-values are \(\pm \frac{\sqrt{3}}{2}\).

The problem asks for the equations of the lines. They are

\[
y = \pm \frac{\sqrt{3}}{2} x = \pm \sqrt{3} x.
\]
4. (15 Points) Coffee is dripping from a conical filter into a cylindrical coffeepot. Both the filter cone and coffeepot have diameter 6 inches. The filter cone has height 6 inches as well. When the coffee in the cone is 5 inches deep, the coffee is draining at the rate of 10 in$^3$/min.

(a) How fast is the level in the pot rising?

*Hint.* The volume of a cylinder of radius $r$ and height $h$ is 
$$V = \pi r^2 h.$$

*Solution.* For the cylinder of coffee in the pot we have 
$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt},$$

($r$ is constant). At this particular moment we have

$$10 = \pi \cdot 6^2 \cdot \frac{dh}{dt},$$

$$\frac{dh}{dt} = \frac{10}{36 \pi} = \frac{5}{18 \pi} \text{ in/min}.$$ 

(b) How fast is the level in the cone falling at the same time?

*Hint.* The volume of a cone of radius $r$ and height $h$ is 
$$V = \frac{1}{3} \pi r^2 h.$$

*Solution.* For the filter cone, $\frac{r}{h} = \frac{1}{2}$. Since the cone of liquid coffee must be similar, we have $r = \frac{h}{2}$ so

$$V = \frac{1}{3} \pi \left(\frac{h}{2}\right)^2 h = \frac{1}{12} \pi h^3.$$

Thus,

$$\frac{dV}{dt} = \frac{1}{4} \pi h^2 \frac{dh}{dt}.$$ 

At this particular moment we have

$$10 = \frac{1}{4} \pi \cdot 5^2 \cdot \frac{dh}{dt} \implies \frac{dh}{dt} = \frac{8}{5 \pi} \text{ in/min}.$$ 

☐
5. (25 Points) We are going to graph completely the function

\[ f(x) = \frac{(x+1)^2}{x^2 + 1}. \]

(a) Find the domain of \( f \).

Solution. (2 points) The square root is always nonnegative, so the denominator is always positive. Hence the domain of the function is \( \mathbb{R} \).

(b) Find the places where \( f \) is positive, negative, or zero.

Solution. (4 points) \( f(x) \) is zero when one of the factors in the numerator is zero, i.e., when \( x = -1 \). Otherwise the function is positive.

Positive on: \((−∞, −1), (−1, ∞)\)

Negative on: \(∅ \) (nowhere)

Zero at: 0

(c) Find all horizontal and vertical asymptotes (if any) of the graph of \( f \).

Solution. (2 points) We have

\[ \lim_{x \to ∞} \frac{(x+1)^2}{x^2 + 1} = 1. \]

So there is a horizontal asymptote at \( y = 1 \). (The limit as \( x \to −∞ \) is also 1.) Since \( f \) is continuous everywhere, there are no vertical asymptotes.

(d) The derivative of \( f \) is

\[ f'(x) = \frac{2(1 - x^2)}{(x^2 + 1)^2}. \]

(This is free information; you do not need to calculate \( f' \).) Find the intervals of increase or decrease.

Solution. (4 points) The function has critical points at \( ±1 \) where \( f'(x) = 0 \). We can make a sign chart:

<table>
<thead>
<tr>
<th></th>
<th>( x &lt; -1 )</th>
<th>(-1 &lt; x &lt; 1 )</th>
<th>( x &gt; 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 - x )</td>
<td>+</td>
<td>+</td>
<td>−</td>
</tr>
<tr>
<td>( 1 + x )</td>
<td>−</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( f'(x) )</td>
<td>( \frac{2(1 - x^2)}{(x^2 + 1)^2} )</td>
<td>−</td>
<td>+</td>
</tr>
<tr>
<td>( f )</td>
<td>( \uparrow )</td>
<td>( \uparrow )</td>
<td>( \downarrow )</td>
</tr>
</tbody>
</table>
Increasing on: \((-1, 1)\)

Decreasing on: \((-\infty, -1], [1, \infty)\)

(e) Find any local maxima or minima.

Solution. (2 points) There is a local maximum at \(x = 1\) (where \(f\) changes from increasing to decreasing) and a local minimum at \(x = -1\) (where \(f\) changes from decreasing to increasing).

(f) The second derivative of \(f\) is

\[ f''(x) = \frac{4x(x^2 - 3)}{(x^2 + 1)^3}. \]

Find the intervals of concavity.

Solution. (4 points) The points of interest are \(x = 0\) and \(x = \pm\sqrt{3}\) where \(f''(x) = 0\). The sign chart is

<table>
<thead>
<tr>
<th>(x)</th>
<th>(-\sqrt{3})</th>
<th>(-\sqrt{3} &lt; x &lt; 0)</th>
<th>(0 &lt; x &lt; \sqrt{3})</th>
<th>(x &gt; 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x^2 - 3)</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>(f'' = \frac{4x(x^2 - 3)}{(x^2 + 1)^3})</td>
<td>--</td>
<td>++</td>
<td>--</td>
<td>++</td>
</tr>
<tr>
<td>(f)</td>
<td>(\circ)</td>
<td>(\circ)</td>
<td>(\circ)</td>
<td>(\circ)</td>
</tr>
</tbody>
</table>

Concave up on: \((-\sqrt{3}, 0), (\sqrt{3}, \infty)\)

Concave down on: \((-\infty, -\sqrt{3}), (0, \sqrt{3})\)

(g) Find any inflection point(s).

Solution. (2 points) \(f\) changes concavity at each critical point of \(f'\): at \(x = 0\) and \(\pm\sqrt{2}\).

(h) Sketch the graph of \(f\). Label all significant data—intercepts, asymptotes, local extrema, and inflection points—with \(x\) and \(y\) coordinates.

Solution. (4 points) We put all the information from (a)–(g) into a chart:
(i) Find the global minimum and maximum, if they exist.

Solution. (1 point) The global minimum is \((-1, 0)\) and the global maximum is \((1, 2)\).
Newbury Comics is selling Atomic County, the new graphic novel by Seth Cohen, and sales are good. Each lot order of comics from the distributor costs money to the store, so they would like to find the number of items to order each time that will minimize the cost of inventory control.

Suppose the store sells 1000 books per week, and each book costs $10 to the distributor, with a $20 surcharge per order for delivery expenses. To keep a single book in the store’s inventory for a week costs $1. Then the average weekly cost of inventory if books are ordered \( q \) at a time is

\[
A(q) = \left( \frac{20}{\text{order}} \right) \left( \frac{1000 \text{ items/week}}{q \text{ items/order}} \right) + \left( \frac{10}{\text{item}} \right) \left( \frac{1000 \text{ items/week}}{\text{week}} \right) + \left( \frac{1}{\text{item } \times \text{ week}} \right) \left( \frac{q \text{ items}}{2} \right)
\]

Without the units, that is

\[
A(q) = \frac{20,000}{q} + 10,000 + \frac{q}{2}
\]

(a) What order quantity \( q \) will minimize the inventory costs? Be sure to check that it is a minimum and not a maximum!

Solution. (8 points) We have

\[
A'(q) = -\frac{20,000}{q^2} + \frac{1}{2}.
\]

This is zero when

\[
\frac{20,000}{q^2} = \frac{1}{2} \implies q^2 = 40,000 \implies q = 200.
\]

Notice also that \( A''(q) = \frac{40,000}{q^3} \), which is positive for all \( q > 0 \). Hence 200 is the lot size that minimizes inventory costs.

(b) Suppose the distributor decides to increase its order charges by an additional $0.10 per item, so the cost of ordering \( q \) items is now $20 + $0.10q.

(i) What is the new cost of inventory function similar to \( A(q) \) above?

Solution. (4 points) In this case we have an inventory-cost function

\[
\tilde{A}(q) = \left( \frac{20 + 0.1q}{q} \right) (1000) + 10,000 + \frac{q}{2},
\]

\[
= A(q) + 100.
\]

(ii) What is the new cost-minimizing lot size?

Solution. (3 points) Since \( A \) and \( \tilde{A} \) differ by a constant, they have the same minimum point.
7. (20 Points) Compute the following integrals. For definite integrals, your answer should be a number. For indefinite integrals, your answer should be the most general antiderivative.

(i) \( \int_{-1}^{1} (3x^2 - 4x + 7) \, dx \)

Solution. (4 points) We have
\[
\int_{-1}^{1} (3x^2 - 4x + 7) \, dx = \left[ x^3 - 2x^2 + 7x \right]_{-1}^{1} = (1 - 2 \cdot 1 + 7 \cdot 1) - (-1 - 2 \cdot (-1) + 7(-1)) = 6 - (-10) = 16.
\]

(ii) \( \int \frac{dt}{t^{\sqrt{t}}} \)

Solution. (4 points)
\[
\int \frac{dt}{t^{\sqrt{t}}} = \int t^{-3/2} \, dt = -2t^{-1/2} + C = \frac{-2}{\sqrt{t}} + C.
\]

(iii) \( \int \sqrt{t} \sin(2^{3/2}) \, dt \)

Solution. (6 points) Let \( u = 2^{3/2} \), so that \( du = 3t^{1/2} \, dt \). Then
\[
\int \sqrt{t} \sin(2^{3/2}) \, dt = \frac{1}{3} \int \sin u \, du = \frac{1}{3} \cos u + C = \frac{1}{3} \cos \left(2^{3/2}\right) + C.
\]

(iv) \( \int_0^{1/2} x^3 (1 + 9x^4)^{-3/2} \, dx \)

Solution. (6 points) Let \( u = 1 + 9x^4 \), so that \( du = 36x^3 \, dx \). The new limits of integration are \( u(0) = 1 \) and \( u(1/2) = 1 + 9 \left(\frac{1}{2}\right)^4 = 1 + \frac{1}{16} = \frac{25}{16} \).
The integral becomes

\[ \int_{0}^{1/2} x^3 (1 + 9x^4)^{-3/2} \, dx = \frac{1}{36} \int_{1}^{25/16} u^{-3/2} \, du \]

\[ = -\frac{1}{18} u^{-1/2} \bigg|_{1}^{25/16} \]

\[ = -\frac{1}{18} \left[ \frac{4}{5} - 1 \right] = \frac{1}{18} \cdot \frac{1}{5} = \frac{1}{90}. \]
8. (10 Points) Consider a parabolic arch whose height and width are the same length $h$.* What is the area enclosed by the arch?

Hint. If you put the origin in the middle of the bottom of the arch, the curve can be described by

$$y = h - ax^2.$$ 

What is $a$?

Solution. The function above must evaluate to zero at $\pm \frac{h}{2}$, so

$$0 = h - a\left(\frac{h}{2}\right)^2 = h - \frac{ah^2}{4} \implies a = \frac{4}{h}.$$ 

Therefore

$$A = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(h - \frac{4}{h}x^2\right) \, dx = 2 \int_0^{\frac{h}{2}} \left(h - \frac{4}{h}x^2\right) \, dx$$

$$= 2 \left[ hx - \frac{4}{3h}x^3 \right]_0^{\frac{h}{2}}$$

$$= 2 \left( \frac{h^2}{2} - \frac{4h^2}{24h} \right) = 2h^2 \left( \frac{1}{2} - \frac{1}{6} \right) = \frac{2}{3} h^2$$

*The Jefferson National Expansion Memorial in St. Louis has height and width 630 feet. Unfortunately (for us) its shape is not a parabolic arch, but a catenary, which has the form

$$y = a(e^x + e^{-x}) + b.$$ 

This makes the area much harder to calculate.
9. (20 Points) Let \( a \) be the function defined by
\[
a(y) = \int_0^y \frac{dt}{\sqrt{1 + t^2}}.
\]  
(Do not try to evaluate the integral in closed form; it is possible but that is not the point.)

(a) Find \( a'(y) \)

Solution. By the First Fundamental Theorem of Calculus we have
\[
a'(y) = \frac{1}{\sqrt{1 + y^2}}.
\]  

Let \( s(x) \) be the inverse function of \( a \), so that
\[
a(s(x)) = x.
\]  
(Again, do not try to invert the expression in closed form; treat this equation as defining \( s \).)

(b) Use (\( * \)) and (\( \dagger \)) to show that
\[
s'(x) = \sqrt{1 + s(x)^2}.
\]

Solution. (8 points) Differentiating (\( \dagger \)), we have
\[
1 = (a \circ s)'(x) = a'(s(x))s'(x)
\]
\[
s'(x) = \frac{1}{a'(s(x))} = \frac{1}{\sqrt{1 + s(x)^2}} = \sqrt{1 + s(x)^2}.
\]

(c) Show that
\[
s''(x) = s(x).
\]

Solution. (7 points) We just differentiate the equation for \( s' \):
\[
s''(x) = \frac{2s(x)s'(x)}{2\sqrt{1 + s(x)^2}} = s(x).
\]