Areas and Lengths in Polar Coordinates

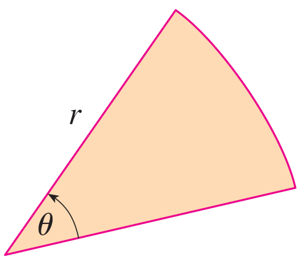
In this section we develop the formula for the area of a region whose boundary is given by a polar equation. We need to use the formula for the area of a sector of a circle

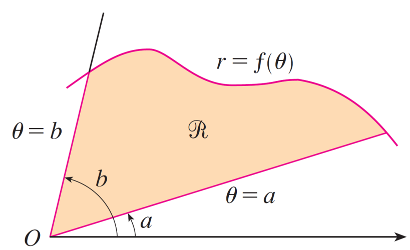
$$\boxed{A = \frac{1}{2}r^2\theta} \tag{1}$$

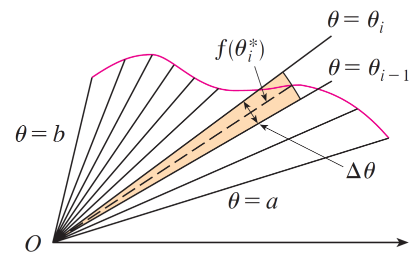
where r is the radius and θ is the radian measure of the central angle. Formula 1 follows from the fact that the area of a sector is proportional to its central angle:

$$A = \frac{\theta}{2\pi}\cdot \pi r^2 = \frac{1}{2}r^2\theta$$

Let $\mathcal{R}$ be the region bounded by the polar curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$, where f is a positive continuous function and where $0 < b - a \leq 2\pi$.







We divide the interval $[a, b]$ into subintervals with endpoints $\theta_0, \theta_1, \theta_2, \ldots, \theta_n$ and equal width $\Delta\theta$. The rays $\theta = \theta_i$ then divide $\mathcal{R}$ into n smaller regions with central angle $\Delta\theta = \theta_i - \theta_{i-1}$. If we choose θ_i^* in the ith subinterval $[\theta_{i-1}, \theta_i]$, then the area ΔA_i of the ith region is approximated by the area of the sector of a circle with central angle $\Delta\theta$ and radius $f(\theta_i^*)$. Thus from Formula 1 we have

$$\Delta A_i \approx \frac{1}{2}[f(\theta_i^*)]^2\Delta\theta \tag{2}$$

and so an approximation to the total area A of $\mathcal{R}$ is $A \approx \sum\limits_{i=1}^{n}\frac{1}{2}[f(\theta_i^*)]^2\Delta\theta$. One can see that the approximation in (2) improves as $n \to \infty$. But the sums in (2) are Riemann sums for the function $g(\theta) = \frac{1}{2}[f(\theta)]^2$, so

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2}[f(\theta_i^*)]^2\Delta\theta = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta$$

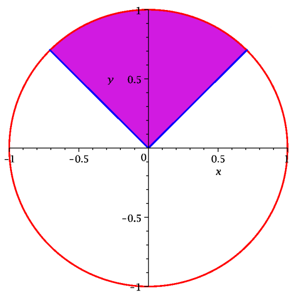
It therefore appears plausible (and can in fact be proved) that the formula for the area A of the polar region $\mathcal{R}$ is

$$\boxed{A = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta} \tag{3}$$

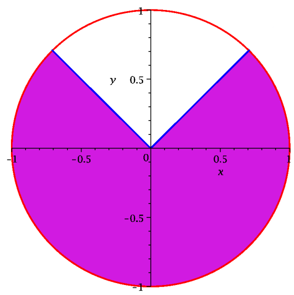
This formula is often written as

$$\boxed{A = \int_a^b \frac{1}{2}r^2 d\theta} \tag{4}$$

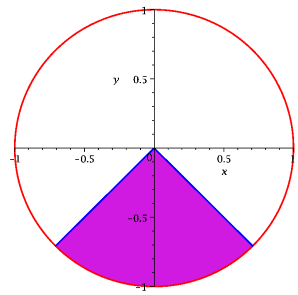
EXAMPLE: Find the area of each of the following regions:



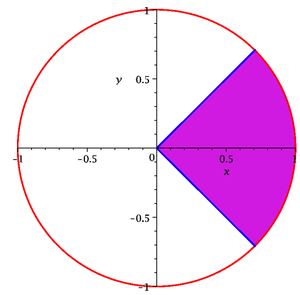
(a)



(b)



(c)



(d)

Solution:

(a) We have

$$A = \int_{\pi/4}^{3\pi/4} \frac{1}{2} \cdot 1^2 d\theta = \frac{1}{2} \int_{\pi/4}^{3\pi/4} d\theta = \frac{1}{2}\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) = \frac{1}{2}\left(\frac{2\pi}{4}\right) = \frac{\pi}{4}$$

(b) We have

$$A = \int_{3\pi/4}^{2\pi+\pi/4} \frac{1}{2} \cdot 1^2 d\theta = \frac{1}{2} \int_{3\pi/4}^{2\pi+\pi/4} d\theta = \frac{1}{2}\left(2\pi + \frac{\pi}{4} - \frac{3\pi}{4}\right) = \frac{1}{2}\left(2\pi - \frac{2\pi}{4}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

(c) We have

$$A = \int_{5\pi/4}^{7\pi/4} \frac{1}{2} \cdot 1^2 d\theta = \frac{1}{2} \int_{5\pi/4}^{7\pi/4} d\theta = \frac{1}{2}\left(\frac{7\pi}{4} - \frac{5\pi}{4}\right) = \frac{1}{2}\left(\frac{2\pi}{4}\right) = \frac{\pi}{4}$$

(d) We have

$$A = \int_{7\pi/4}^{2\pi+\pi/4} \frac{1}{2} \cdot 1^2 d\theta = \frac{1}{2} \int_{7\pi/4}^{2\pi+\pi/4} d\theta = \frac{1}{2}\left(2\pi + \frac{\pi}{4} - \frac{7\pi}{4}\right) = \frac{1}{2}\left(2\pi - \frac{6\pi}{4}\right) = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$$

or

$$A = \int_{-\pi/4}^{\pi/4} \frac{1}{2} \cdot 1^2 d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} d\theta = \frac{1}{2}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \frac{1}{2}\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \frac{\pi}{4}$$

EXAMPLE: Find the area of the inner loop of $r = 2 + 4\cos\theta$.

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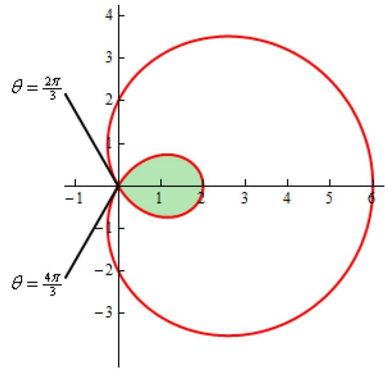

Solution: We first find a and b:

$$2 + 4\cos\theta = 0 \quad \Longrightarrow \quad \cos\theta = -\frac{1}{2} \quad \Longrightarrow \quad \theta = \frac{2\pi}{3}, \ \frac{4\pi}{3}$$

Therefore the area is

$$\begin{aligned}
A &= \int_{2\pi/3}^{4\pi/3} \frac{1}{2}(2 + 4\cos\theta)^2 d\theta = \int_{2\pi/3}^{4\pi/3} \frac{1}{2}(4 + 16\cos\theta + 16\cos^2\theta) d\theta \\
&= \int_{2\pi/3}^{4\pi/3} (2 + 8\cos\theta + 8\cos^2\theta) d\theta = \int_{2\pi/3}^{4\pi/3} \left(2 + 8\cos\theta + 8 \cdot \frac{1 + \cos 2\theta}{2}\right) d\theta \\
&= \int_{2\pi/3}^{4\pi/3} (2 + 8\cos\theta + 4(1 + \cos 2\theta)) d\theta = \int_{2\pi/3}^{4\pi/3} (6 + 8\cos\theta + 4\cos 2\theta) d\theta \\
&= \Big[6\theta + 8\sin\theta + 2\sin 2\theta\Big]_{2\pi/3}^{4\pi/3} = 4\pi - 6\sqrt{3} \approx 2.174
\end{aligned}$$

EXAMPLE: Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

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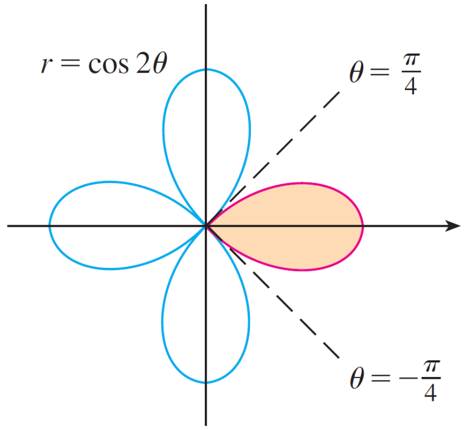
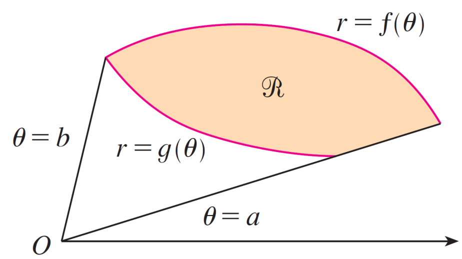


Solution: Notice that the region enclosed by the right loop is swept out by a ray that rotates from $\theta = -\pi/4$ to $\theta = \pi/4$. Therefore, Formula 4 gives

$$\begin{aligned} A &= \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta d\theta = \int_0^{\pi/4} \cos^2 2\theta d\theta \\ &= \int_0^{\pi/4} \frac{1}{2}(1+\cos 4\theta) d\theta = \frac{1}{2}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = \frac{\pi}{8} \end{aligned}$$

Let $\mathcal{R}$ be the region bounded by curves with polar equations $r = f(\theta)$, $r = g(\theta)$, $\theta = a$, and $\theta = b$, where $f(\theta) \geq g(\theta) \geq 0$ and $0 < b - a \leq 2\pi$. Then the area A of $\mathcal{R}$ is

$$\boxed{A = \int_a^b \frac{1}{2}\left([f(\theta)]^2 - [g(\theta)]^2\right) d\theta}$$



EXAMPLE: Find the area that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.

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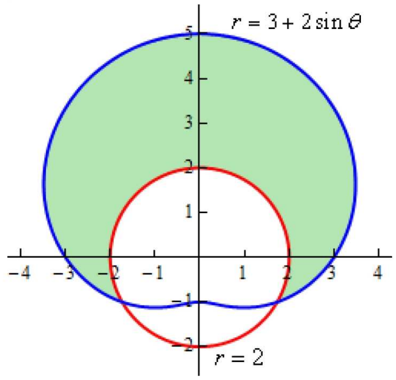

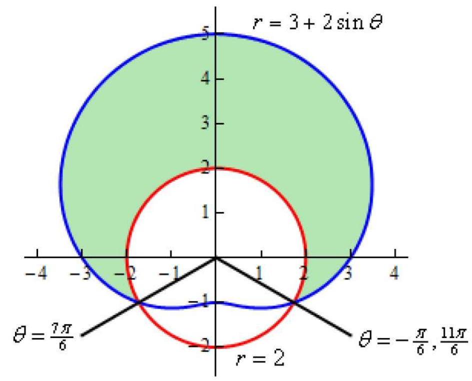

Solution: We first find a and b:

$$3 + 2\sin\theta = 2 \quad \Longrightarrow \quad \sin\theta = -\frac{1}{2} \quad \Longrightarrow \quad \theta = \frac{7\pi}{6},\ -\frac{\pi}{6}\left(\frac{11\pi}{6}\right)$$

Therefore the area is

$$\begin{aligned} A &= \int_{-\pi/6}^{7\pi/6} \frac{1}{2}\left[(3 + 2\sin\theta)^2 - 2^2\right] d\theta = \int_{-\pi/6}^{7\pi/6} \frac{1}{2}(5 + 12\sin\theta + 4\sin^2\theta)d\theta \\ &= \int_{-\pi/6}^{7\pi/6} \frac{1}{2}\left(5 + 12\sin\theta + 4\cdot\frac{1-\cos 2\theta}{2}\right) d\theta = \int_{-\pi/6}^{7\pi/6} \frac{1}{2}(5 + 12\sin\theta + 2(1 - \cos 2\theta))d\theta \\ &= \int_{-\pi/6}^{7\pi/6} \frac{1}{2}(7 + 12\sin\theta - 2\cos 2\theta)d\theta = \frac{1}{2}\Big[7\theta - 12\cos\theta - \sin 2\theta\Big]_{-\pi/6}^{7\pi/6} \\ &= \frac{11\sqrt{3}}{2} + \frac{14\pi}{3} \approx 24.187 \end{aligned}$$

EXAMPLE: Find the area of the region outside $r = 3 + 2\sin\theta$ and inside $r = 2$.

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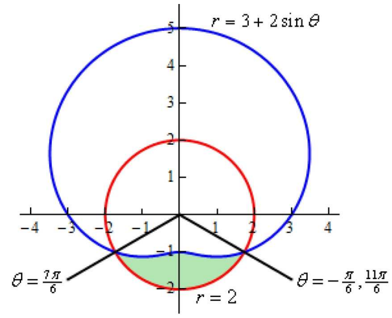
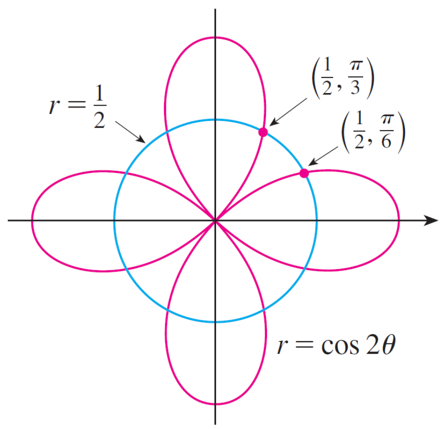
Solution: We have



$$\begin{aligned}A &= \int_{7\pi/6}^{11\pi/6}\frac{1}{2}\left[2^2-(3+2\sin\theta)^2\right]d\theta\\ &= \int_{7\pi/6}^{11\pi/6}\frac{1}{2}(-5-12\sin\theta-4\sin^2\theta)d\theta\\ &= \int_{7\pi/6}^{11\pi/6}\frac{1}{2}(-7-12\sin\theta+2\cos 2\theta)d\theta=\frac{1}{2}\Big[-7\theta+12\cos\theta+\sin 2\theta\Big]_{7\pi/6}^{11\pi/6}=\frac{11\sqrt{3}}{2}-\frac{7\pi}{3}\approx 2.196\end{aligned}$$

EXAMPLE: Find all points of intersection of the curves $r=\cos 2\theta$ and $r=\dfrac{1}{2}$.



Solution: If we solve the equations $r=\cos 2\theta$ and $r=\dfrac{1}{2}$, we get $\cos 2\theta=\dfrac{1}{2}$ and, therefore,

$$2\theta=\pi/3,\ 5\pi/3,\ 7\pi/3,\ 11\pi/3$$

Thus the values of θ between 0 and 2π that satisfy both equations are

$$\theta=\pi/6,\ 5\pi/6,\ 7\pi/6,\ 11\pi/6$$

We have found four points of intersection:

$$\left(\frac{1}{2},\pi/6\right),\ \left(\frac{1}{2},5\pi/6\right),\ \left(\frac{1}{2},7\pi/6\right),\ \text{and}\ \left(\frac{1}{2},11\pi/6\right)$$

However, you can see from the above figure that the curves have four other points of intersection — namely,

$$\left(\frac{1}{2},\pi/3\right),\ \left(\frac{1}{2},2\pi/3\right),\ \left(\frac{1}{2},4\pi/3\right),\ \text{and}\ \left(\frac{1}{2},5\pi/3\right)$$

These can be found using symmetry or by noticing that another equation of the circle is $r=-\dfrac{1}{2}$ and then solving the equations $r=\cos 2\theta$ and $r=-\dfrac{1}{2}$.

Arc Length

To find the length of a polar curve $r = f(\theta), a \le \theta \le b$, we regard θ as a parameter and write the parametric equations of the curve as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Using the Product Rule and differentiating with respect to θ, we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

So, using $\cos^2\theta + \sin^2\theta = 1$, we have

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta$$

$$+\left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta = \left(\frac{dr}{d\theta}\right)^2 + r^2$$

Assuming that f' is continuous, we can use one of the formulas from Section 9.2 to write the arc length as

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Therefore, the length of a curve with polar equation $r = f(\theta), a \le \theta \le b$, is

$$\boxed{L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta} \qquad (5)$$

EXAMPLE: Find the length of the curve $r = \theta,\ 0 \le \theta \le 1$.

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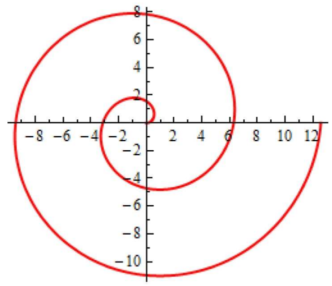

Solution: We have

$$L = \int_0^1 \sqrt{\theta^2+1}d\theta = \left[\begin{array}{l} \theta = \tan x \quad \Longrightarrow \quad \sqrt{\theta^2+1} = \sqrt{\tan^2 x + 1} = \sqrt{\sec^2 x} = |\sec x| = \sec x \\ d\theta = d\tan x \\ d\theta = \sec^2 x dx \end{array}\right]$$

$$= \int_0^{\pi/4} \sec^3 x dx = \left[\frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|)\right]_0^{\pi/4} = \frac{1}{2}(\sqrt{2} + \ln(1+\sqrt{2}))$$

EXAMPLE: Find the length of the cardioid $r = 1 - \cos\theta$.

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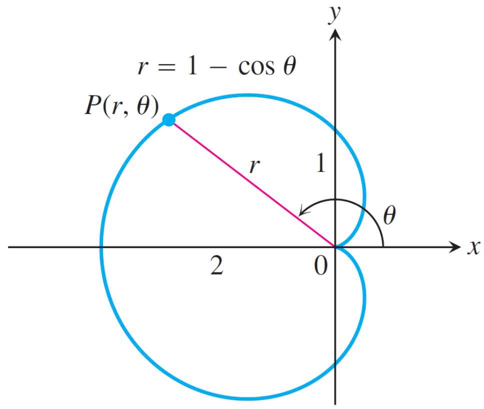


Solution: The full length of the cardioid is given by the parameter interval $0 \le \theta \le 2\pi$, so Formula 5 gives

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_0^{2\pi} \sqrt{(1-\cos\theta)^2 + \sin^2\theta d\theta}$$

$$= \int_0^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta d\theta}$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos\theta d\theta}$$

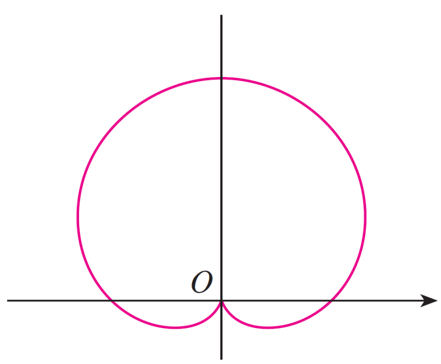
$$= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2} d\theta}$$

$$= \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta$$

$$= \int_0^{2\pi} 2\sin\frac{\theta}{2} d\theta$$

$$= -4\cos\frac{\theta}{2}\Big]_0^{2\pi} = 4 + 4 = 8$$

EXAMPLE: Find the length of the cardioid $r = 1 + \sin\theta$.



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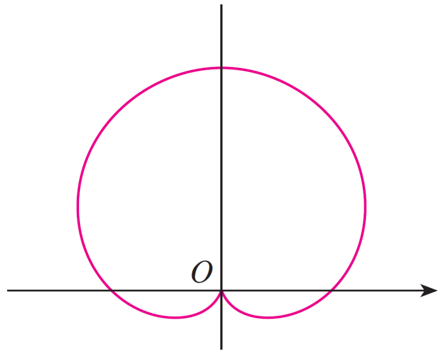

Solution 1: Note that

$$r = 1 + \sin\theta = 1 - \cos\left(\theta + \frac{\pi}{2}\right)$$

Therefore the graph of $r = 1 + \sin\theta$ is the rotation of the graph of $r = 1 - \cos\theta$. Hence the length of the cardioid $r = 1 + \sin\theta$ is 8 by the previous Example.

Solution 2: The full length of the cardioid is given by the parameter interval $0 \le \theta \le 2\pi$, so Formula 5 gives

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

$$= \int_0^{2\pi} \sqrt{(1 + \sin\theta)^2 + \cos^2\theta} d\theta$$

$$= \int_0^{2\pi} \sqrt{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta} d\theta$$

$$= \int_0^{2\pi} \sqrt{2 + 2\sin\theta} d\theta$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos\left(\theta + \frac{\pi}{2}\right)} d\theta = \begin{bmatrix} \theta + \frac{\pi}{2} = u \\ d\left(\theta + \frac{\pi}{2}\right) = du \\ d\theta = du \end{bmatrix} = \int_{\pi/2}^{5\pi/2} \sqrt{2 - 2\cos u} du$$

$$= \int_{\pi/2}^{5\pi/2} \sqrt{4\sin^2\frac{u}{2}} du = \int_{\pi/2}^{5\pi/2} 2\left|\sin\frac{u}{2}\right| du = \int_{\pi/2}^{2\pi} 2\left|\sin\frac{u}{2}\right| du + \int_{2\pi}^{5\pi/2} 2\left|\sin\frac{u}{2}\right| du$$

$$= \int_{\pi/2}^{2\pi} 2\sin\frac{u}{2} du - \int_{2\pi}^{5\pi/2} 2\sin\frac{u}{2} du$$

$$= -4\cos\frac{u}{2}\bigg]_{\pi/2}^{2\pi} + 4\cos\frac{u}{2}\bigg]_{2\pi}^{5\pi/2}$$

$$= \left(-4\cos\pi + 4\cos\frac{\pi}{4}\right) + \left(4\cos\frac{5\pi}{4} - 4\cos\pi\right)$$

$$= (4 + 2\sqrt{2}) + (-2\sqrt{2} + 4) = 8$$

Solution 3: The full length of the cardioid is given by the parameter interval $0 \le \theta \le 2\pi$, so Formula 5 gives

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta} d\theta = \int_0^{2\pi} \sqrt{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta} d\theta \\
&= \int_0^{2\pi} \sqrt{2 + 2\sin\theta} d\theta = \sqrt{2}\int_0^{2\pi} \sqrt{1+\sin\theta} d\theta = \sqrt{2}\int_0^{2\pi} \frac{\sqrt{1+\sin\theta}\sqrt{1-\sin\theta}}{\sqrt{1-\sin\theta}} d\theta \\
&= \sqrt{2}\int_0^{2\pi} \frac{\sqrt{1-\sin^2\theta}}{\sqrt{1-\sin\theta}} d\theta = \sqrt{2}\int_0^{2\pi} \frac{\sqrt{\cos^2\theta}}{\sqrt{1-\sin\theta}} d\theta = \sqrt{2}\int_0^{2\pi} \frac{|\cos\theta|}{\sqrt{1-\sin\theta}} d\theta \\
&= \sqrt{2}\int_0^{\pi/2} \frac{\cos\theta}{\sqrt{1-\sin\theta}} d\theta - \sqrt{2}\int_{\pi/2}^{3\pi/2} \frac{\cos\theta}{\sqrt{1-\sin\theta}} d\theta + \sqrt{2}\int_{3\pi/2}^{2\pi} \frac{\cos\theta}{\sqrt{1-\sin\theta}} d\theta
\end{aligned}$$

Note that

$$\int \frac{\cos\theta}{\sqrt{1-\sin\theta}} d\theta = \begin{bmatrix} 1-\sin\theta = u \\ d(1-\sin\theta) = du \\ -\cos\theta d\theta = du \\ \cos\theta d\theta = -du \end{bmatrix} = -\int \frac{du}{\sqrt{u}} = -\int u^{-1/2} du = -\frac{u^{-1/2+1}}{-1/2+1} + C$$

$$= -2\sqrt{u} + C$$

$$= -2\sqrt{1-\sin\theta} + C$$

Therefore

$$\begin{aligned}
L &= -2\sqrt{2}\sqrt{1-\sin\theta}\Big]_0^{\pi/2} + 2\sqrt{2}\sqrt{1-\sin\theta}\Big]_{\pi/2}^{3\pi/2} - 2\sqrt{2}\sqrt{1-\sin\theta}\Big]_{3\pi/2}^{2\pi} \\
&= -2\sqrt{2}\left(\sqrt{1-\sin(\pi/2)} - \sqrt{1-\sin 0}\right) \\
&\quad +2\sqrt{2}\left(\sqrt{1-\sin(3\pi/2)} - \sqrt{1-\sin(\pi/2)}\right) \\
&\quad -2\sqrt{2}\left(\sqrt{1-\sin(2\pi)} - \sqrt{1-\sin(3\pi/2)}\right) \\
&= -2\sqrt{2}\,(0-1) + 2\sqrt{2}\left(\sqrt{2}-0\right) - 2\sqrt{2}\left(1-\sqrt{2}\right) \\
&= 2\sqrt{2} + 4 - 2\sqrt{2} + 4 = 8
\end{aligned}$$