Calculus with Parametric Curves

Suppose f and g are differentiable functions and we want to find the tangent line at a point on the parametric curve $x = f(t),\ y = g(t)$ where y is also a differentiable function of x. Then the Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

If $dx/dt \neq 0$, we can solve for dy/dx:

$$\boxed{\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0} \tag{1}$$

It can be seen from (1) that the curve has a horizontal tangent when $dy/dt = 0$ (provided that $dx/dt \neq 0$) and it has a vertical tangent when $dx/dt = 0$ (provided that $dy/dt \neq 0$).

EXAMPLE 1: A curve C is defined by the parametric equations $x = t^2,\ y = t^3 - 3t$.

(a) Show that C has two tangents at the point $(3, 0)$ and find their equations.

(b) Find the points on C where the tangent is horizontal or vertical.

(c) Determine where the curve is concave upward or downward.

(d) Sketch the curve.

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(d) Sketch the curve.

Solution:

(a) Notice that $x = t^2 = 3$ when $t = \pm\sqrt{3}$. Therefore, the point (3, 0) on C arises from two values of the parameter, $t = \sqrt{3}$ and $t = -\sqrt{3}$. This indicates that the curve C crosses itself at (3, 0). Since

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right)$$

the slope of the tangent when $t = \pm\sqrt{3}$ is $dy/dx = \pm 6/(2\sqrt{3}) = \pm\sqrt{3}$, so the equations of the tangents at (3, 0) are

$$y = \sqrt{3}(x - 3) \quad \text{and} \quad y = -\sqrt{3}(x - 3)$$

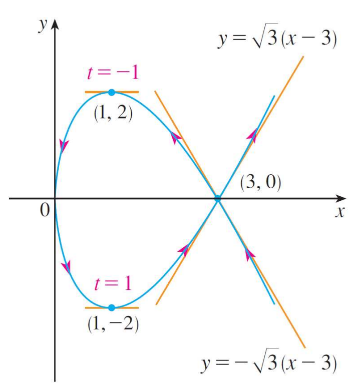
(b) C has a horizontal tangent when $dy/dx = 0$, that is, when $dy/dt = 0$ and $dx/dt \neq 0$. Since $dy/dt = 3t^2 - 3$, this happens when $t^2 = 1$, that is, $t = \pm 1$. The corresponding points on C are $(1, -2)$ and $(1, 2)$. C has a vertical tangent when $dx/dt = 2t = 0$, that is, $t = 0$. (Note that $dy/dt \neq 0$ there.) The corresponding point on C is $(0, 0)$.

(c) To determine concavity we calculate the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}} = \frac{\dfrac{3}{2}\left(1 + \dfrac{1}{t^2}\right)}{2t} = \frac{3(t^2 + 1)}{4t^3}$$

Thus the curve is concave upward when $t > 0$ and concave downward when $t < 0$.

(d) Using the information from parts (b) and (c), we sketch C:



EXAMPLE 2:

(a) Find the tangent to the cycloid $x = r(\theta - \sin\theta),\ y = r(1 - \cos\theta)$ at the point where $\theta = \pi/3$.

(b) At what points is the tangent horizontal? When is it vertical?

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Solution:

(a) The slope of the tangent line is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\sin\theta}{r(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$$

When $\theta = \pi/3$, we have

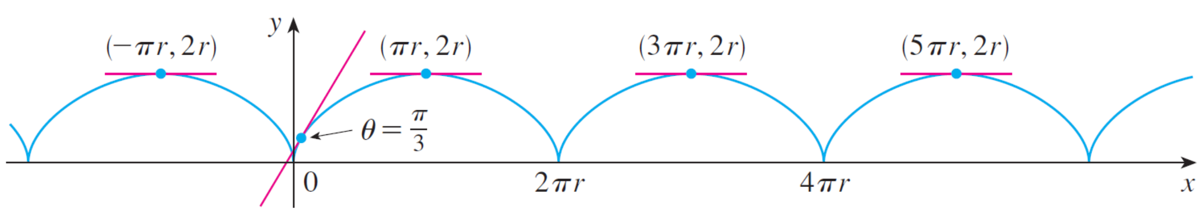
$$x = r\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right) = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right), \quad y = r\left(1 - \cos\frac{\pi}{3}\right) = \frac{r}{2}$$

and

$$\frac{dy}{dx} = \frac{\sin(\pi/3)}{1-\cos(\pi/3)} = \frac{\sqrt{3}/2}{1-\frac{1}{2}} = \sqrt{3}$$

Therefore the slope of the tangent is $\sqrt{3}$ and its equation is

$$y - \frac{r}{2} = \sqrt{3}\left(x - \frac{r\pi}{3} + \frac{r\sqrt{3}}{2}\right) \quad \text{or} \quad \sqrt{3}x - y = r\left(\frac{\pi}{\sqrt{3}} - 2\right)$$



(b) The tangent is horizontal when $dy/dx = 0$, which occurs when $\sin\theta = 0$ and $1 - \cos\theta \neq 0$, that is, $\theta = (2n-1)\pi$, n an integer. The corresponding point on the cycloid is $((2n-1)\pi r,\ 2r)$.

When $\theta = 2n\pi$, both $dx/d\theta$ and $dy/d\theta$ are 0. It appears from the graph that there are vertical tangents at those points. We can verify this by using l'Hospital's Rule as follows:

$$\lim_{\theta\to 2n\pi^+} \frac{dy}{dx} = \lim_{\theta\to 2n\pi^+} \frac{\sin\theta}{1-\cos\theta} = \lim_{\theta\to 2n\pi^+} \frac{\cos\theta}{\sin\theta} = \infty$$

A similar computation shows that $dy/dx \to -\infty$ as $\theta \to 2n\pi^-$, so indeed there are vertical tangents when $\theta = 2n\pi$, that is, when $x = 2n\pi r$.

Areas

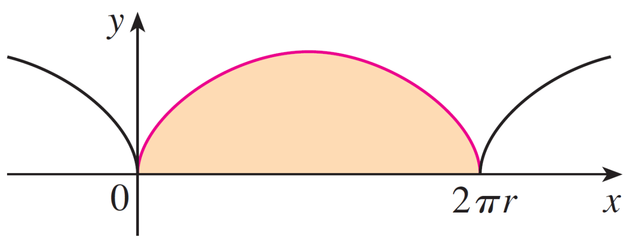
We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)dx$, where $F(x) \geq 0$. If the curve is given by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

$$\boxed{A = \int_a^b y dx = \int_\alpha^\beta g(t) f'(t) dt}$$

$$\left[\text{or } \int_\beta^\alpha g(t)f'(t)dt \ \text{ if } \ (f(\beta), g(\beta)) \text{ is the leftmost endpoint}\right]$$

EXAMPLE 3: Find the area under one arch of the cycloid

$$x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta)$$



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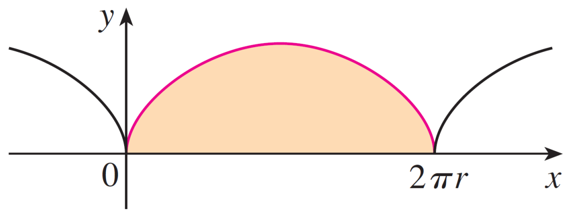


Solution: One arch of the cycloid is given by $0 \le \theta \le 2\pi$. Using the Substitution Rule with $y = r(1-\cos\theta)$ and $dx = r(1-\cos\theta)d\theta$, we have

$$\begin{aligned} A = \int_0^{2\pi r} y dx &= \int_0^{2\pi} r(1-\cos\theta)r(1-\cos\theta)d\theta \\ &= r^2 \int_0^{2\pi} (1-\cos\theta)^2 d\theta \\ &= r^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)d\theta \\ &= r^2 \int_0^{2\pi} \left[1 - 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right] d\theta \\ &= r^2 \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} \\ &= r^2 \left(\frac{3}{2}\cdot 2\pi\right) = 3\pi r^2 \end{aligned}$$

Arc Length

We already know how to find the length L of a curve C given in the form $y = F(x)$, $a \le x \le b$. In fact, if F' is continuous, then

$$\boxed{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx} \tag{2}$$

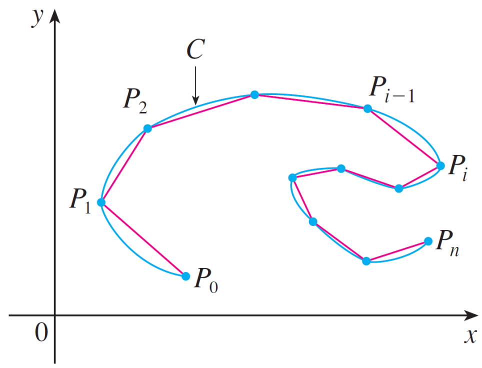
Suppose that C can also be described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \le t \le \beta$, where $dx/dt = f'(t) > 0$. This means that C is traversed once, from left to right, as t increases from α to β and $f(\alpha) = a$, $f(\beta) = b$. Putting Formula 1 into Formula 2 and using the Substitution Rule, we obtain

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt} dt$$

Since $dx/dt > 0$, we have

$$\boxed{L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt} \tag{3}$$

Even if C can't be expressed in the form $y = F(x)$, Formula 3 is still valid but we obtain it by polygonal approximations. We divide the parameter interval $[\alpha, \beta]$ into n subintervals of equal width Δt. If $t_0, t_1, t_2, \ldots, t_n$ are the endpoints of these subintervals, then $x_i = f(t_i)$ and $y_i = g(t_i)$ are the coordinates of points $P_i(x_i,\ y_i)$ that lie on C and the polygon with vertices $P_0, P_1, \ldots, P_n$ approximates C.



As in Section 7.4, we define the length L of C to be the limit of the lengths of these approximating polygons as $n \to \infty$:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

The Mean Value Theorem, when applied to f on the interval $[t_{i-1},\ t_i]$, gives a number t_i^* in $(t_{i-1},\ t_i)$ such that

$$f(t_i) - f(t_{i-1}) = f'(t_i^*)(t_i - t_{i-1})$$

If we let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, this equation becomes

$$\Delta x_i = f'(t_i^*)\Delta t$$

Similarly, when applied to g, the Mean Value Theorem gives a number t_i^{**} in $(t_{i-1},\ t_i)$ such that

$$\Delta y_i = g'(t_i^{**})\Delta t$$

Therefore

$$|P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{[f'(t_i^*)\Delta t]^2 + [g'(t_i^{**})\Delta t]^2} = \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\Delta t$$

and so

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\Delta t \tag{4}$$

The sum in (4) resembles a Riemann sum for the function $\sqrt{[f'(t)]^2 + [g'(t)]^2}$ but it is not exactly a Riemann sum because $t_i^* \neq t_i^{**}$ in general. Nevertheless, if f' and g' are continuous, it can be shown that the limit in (4) is the same as if t_i^* and t_i^{**} were equal, namely,

$$L = \int_{\alpha}^{\beta} \sqrt{[f'(t)]^2 + [g'(t)]^2}dt$$

Thus, using Leibniz notation, we have the following result, which has the same form as (3).

THEOREM: If a curve C is described by the parametric equations $x = f(t)$, $y = g(t), \alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$\boxed{L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Notice that the formula in this Theorem is consistent with the general formulas

$$L = \int ds \quad \text{and} \quad (ds)^2 = (dx)^2 + (dy)^2$$

of Section 7.4.

EXAMPLE 4: If we use the representation of the unit circle

$$x = \cos t, \quad y = \sin t \quad 0 \leq t \leq 2\pi$$

then $dx/dt = -\sin t$ and $dy/dt = \cos t$, so the above Theorem gives

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t} dt = \int_0^{2\pi} dt = 2\pi$$

as expected. If, on the other hand, we use the representation

$$x = \sin 2t, \quad y = \cos 2t \quad 0 \leq t \leq 2\pi$$

then $dx/dt = 2\cos 2t$, $dy/dt = -2\sin 2t$, and the integral in the above Theorem gives

$$\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{4\cos^2 2t + 4\sin^2 2t} dt = \int_0^{2\pi} 2dt = 4\pi$$

REMARK: Notice that the integral gives twice the arc length of the circle because as t increases from 0 to 2π, the point $(\sin 2t, \cos 2t)$ traverses the circle twice. In general, when finding the length of a curve C from a parametric representation, we have to be careful to ensure that C is traversed only once as t increases from α to β.

EXAMPLE 5: Find the length of one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.

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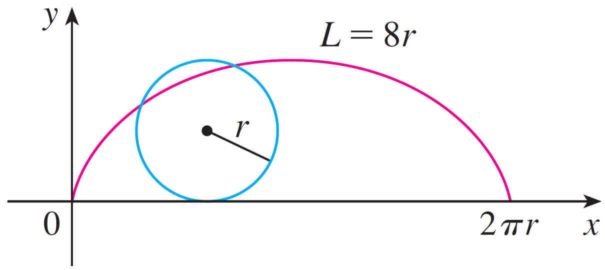

Solution: From Example 3 we see that one arch is described by the parameter interval $0 \le \theta \le 2\pi$. Since

$$\frac{dx}{d\theta} = r(1 - \cos\theta) \quad \text{and} \quad \frac{dy}{d\theta} = r\sin\theta$$

we have

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{r^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)}\, d\theta \\ &= r\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta \end{aligned}$$

To evaluate this integral we use the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ with $\theta = 2x$, which gives

$$1 - \cos\theta = 2\sin^2(\theta/2)$$

Since $0 \le \theta \le 2\pi$, we have $0 \le \theta/2 \le \pi$ and so $\sin(\theta/2) \ge 0$. Therefore

$$\sqrt{2(1-\cos\theta)} = \sqrt{4\sin^2(\theta/2)} = 2|\sin(\theta/2)| = 2\sin(\theta/2)$$

and so

$$L = 2r\int_0^{2\pi} \sin(\theta/2)\, d\theta = 2r[-2\cos(\theta/2)]_0^{2\pi} = 2r[2+2] = 8r$$

EXAMPLE 6: Find the length of the astroid $x = \cos^3 t$, $y = \sin^3 t$, $0 \le t \le 2\pi$.

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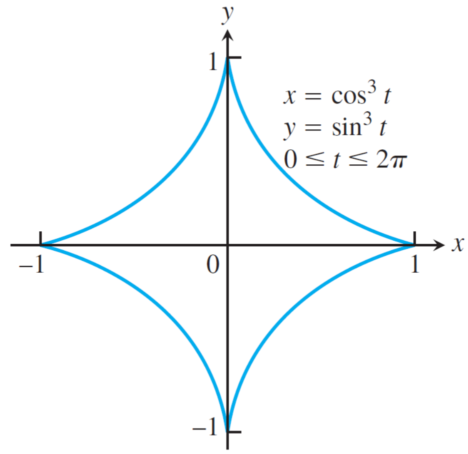

Solution: Since

$$\frac{dx}{dt} = 3\cos^2 t(-\sin t) \quad \text{and} \quad \frac{dy}{dt} = 3\sin^2 t\cos t$$

we have

$$\begin{aligned}
\int_0^{\pi/2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt &= \int_0^{\pi/2}\sqrt{9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t}dt \\
&= \int_0^{\pi/2}\sqrt{9\sin^2 t\cos^2 t(\cos^2 t+\sin^2 t)}dt \\
&= \int_0^{\pi/2}\sqrt{9\sin^2 t\cos^2 t}dt \\
&= \int_0^{\pi/2}|3\sin t\cos t|dt \\
&= \int_0^{\pi/2} 3\sin t\cos t dt = \left[\begin{array}{l}\sin t = u\\ d(\sin t) = du\\ \cos t dt = du\end{array}\right] = \int_0^1 3u du = \left.\frac{3u^2}{2}\right]_0^1 = \frac{3}{2}
\end{aligned}$$

It follows that

$$\text{Length of first-quadrant portion} = \int_0^{\pi/2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt = \frac{3}{2}$$

Therefore the length of the astroid is four times this: $4\cdot\dfrac{3}{2} = 6$.