Suppose $f$ and $g$ are differentiable functions and we want to find the tangent line at a point on the parametric curve $x = f(t), \ y = g(t)$ where $y$ is also a differentiable function of $x$. Then the Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

If $dx/dt \neq 0$, we can solve for $dy/dx$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0$$

It can be seen from (1) that the curve has a horizontal tangent when $dy/dt = 0$ (provided that $dx/dt \neq 0$) and it has a vertical tangent when $dx/dt = 0$ (provided that $dy/dt \neq 0$).

EXAMPLE 1: A curve $C$ is defined by the parametric equations $x = t^2, \ y = t^3 - 3t$.

(a) Show that $C$ has two tangents at the point $(3, 0)$ and find their equations.

(b) Find the points on $C$ where the tangent is horizontal or vertical.

(c) Determine where the curve is concave upward or downward.

(d) Sketch the curve.
EXAMPLE 1: A curve $C$ is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.

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(d) Sketch the curve.

Solution:

(a) Notice that $x = t^2 = 3$ when $t = \pm \sqrt{3}$. Therefore, the point $(3, 0)$ on $C$ arises from two values of the parameter, $t = \sqrt{3}$ and $t = -\sqrt{3}$. This indicates that the curve $C$ crosses itself at $(3, 0)$. Since

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t} = \frac{3}{2} \left( t - \frac{1}{t} \right)$$

the slope of the tangent when $t = \pm \sqrt{3}$ is $dy/dx = \pm 6/(2\sqrt{3}) = \pm \sqrt{3}$, so the equations of the tangents at $(3, 0)$ are

$$y = \sqrt{3}(x - 3) \quad \text{and} \quad y = -\sqrt{3}(x - 3)$$

(b) $C$ has a horizontal tangent when $dy/dx = 0$, that is, when $dy/dt = 0$ and $dx/dt \neq 0$. Since $dy/dt = 3t^2 - 3$, this happens when $t^2 = 1$, that is, $t = \pm 1$. The corresponding points on $C$ are $(1, -2)$ and $(1, 2)$. $C$ has a vertical tangent when $dx/dt = 2t = 0$, that is, $t = 0$. (Note that $dy/dt \neq 0$ there.) The corresponding point on $C$ is $(0, 0)$.

(c) To determine concavity we calculate the second derivative:

$$\frac{d^2y}{dx^2} = \frac{d}{dt} \left( \frac{dy}{dx} \right) = \frac{3}{2} \left( 1 + \frac{1}{t^2} \right) = \frac{3(t^2 + 1)}{4t^3}$$

Thus the curve is concave upward when $t > 0$ and concave downward when $t < 0$.

(d) Using the information from parts (b) and (c), we sketch $C$:

EXAMPLE 2:

(a) Find the tangent to the cycloid $x = r(\theta - \sin \theta)$, $y = r(1 - \cos \theta)$ at the point where $\theta = \pi/3$.

(b) At what points is the tangent horizontal? When is it vertical?
EXAMPLE 2:
(a) Find the tangent to the cycloid \( x = r(\theta - \sin \theta), \ y = r(1 - \cos \theta) \) at the point where \( \theta = \pi/3 \).
(b) At what points is the tangent horizontal? When is it vertical?

Solution:
(a) The slope of the tangent line is

\[
\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{r \sin \theta}{r(1 - \cos \theta)} = \frac{\sin \theta}{1 - \cos \theta}
\]

When \( \theta = \pi/3 \), we have

\[
x = r \left( \frac{\pi}{3} - \sin \frac{\pi}{3} \right) = r \left( \frac{\pi}{3} - \frac{\sqrt{3}}{2} \right), \quad y = r \left( 1 - \cos \frac{\pi}{3} \right) = \frac{r}{2}
\]

and

\[
\frac{dy}{dx} = \frac{\sin(\pi/3)}{1 - \cos(\pi/3)} = \frac{\sqrt{3}/2}{1 - 1/2} = \sqrt{3}
\]

Therefore the slope of the tangent is \( \sqrt{3} \) and its equation is

\[
y - \frac{r}{2} = \sqrt{3} \left( x - \frac{r\pi}{3} + \frac{r\sqrt{3}}{2} \right) \quad \text{or} \quad \sqrt{3}x - y = r \left( \frac{\pi}{\sqrt{3}} - 2 \right)
\]

(b) The tangent is horizontal when \( \frac{dy}{dx} = 0 \), which occurs when \( \sin \theta = 0 \) and \( 1 - \cos \theta \neq 0 \), that is, \( \theta = (2n - 1)\pi, \ n \) an integer. The corresponding point on the cycloid is \((2n - 1)\pi r, 2r\).

When \( \theta = 2n\pi \), both \( dx/d\theta \) and \( dy/d\theta \) are 0. It appears from the graph that there are vertical tangents at those points. We can verify this by using l’Hospital’s Rule as follows:

\[
\lim_{\theta \to 2n\pi^+} \frac{dy}{dx} = \lim_{\theta \to 2n\pi^+} \frac{\sin \theta}{1 - \cos \theta} = \lim_{\theta \to 2n\pi^+} \frac{\cos \theta}{\sin \theta} = \infty
\]

A similar computation shows that \( dy/dx \to -\infty \) as \( \theta \to 2n\pi^- \), so indeed there are vertical tangents when \( \theta = 2n\pi \), that is, when \( x = 2n\pi r \).
Areas

We know that the area under a curve \( y = F(x) \) from \( a \) to \( b \) is \( A = \int_{a}^{b} F(x) \, dx \), where \( F(x) \geq 0 \). If the curve is given by the parametric equations \( x = f(t) \) and \( y = g(t) \), \( \alpha \leq t \leq \beta \), then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

\[
A = \int_{a}^{b} y \, dx = \int_{\alpha}^{\beta} g(t) f'(t) \, dt
\]

[or \( \int_{\beta}^{\alpha} g(t) f'(t) \, dt \) if \((f(\beta), g(\beta))\) is the leftmost endpoint]

EXAMPLE 3: Find the area under one arch of the cycloid

\[
x = r(\theta - \sin \theta), \quad y = r(1 - \cos \theta)
\]
EXAMPLE 3: Find the area under one arch of the cycloid
\[ x = r(\theta - \sin \theta), \quad y = r(1 - \cos \theta) \]

Solution: One arch of the cycloid is given by \(0 \leq \theta \leq 2\pi\). Using the Substitution Rule with \(y = r(1 - \cos \theta)\) and \(dx = r(1 - \cos \theta) d\theta\), we have
\[
A = \int_0^{2\pi} y \, dx = \int_0^{2\pi} r(1 - \cos \theta) r(1 - \cos \theta) \, d\theta
\]
\[
= r^2 \int_0^{2\pi} (1 - \cos \theta)^2 \, d\theta
\]
\[
= r^2 \int_0^{2\pi} (1 - 2 \cos \theta + \cos^2 \theta) \, d\theta
\]
\[
= r^2 \int_0^{2\pi} \left[1 - 2 \cos \theta + \frac{1}{2} (1 + \cos 2\theta)\right] \, d\theta
\]
\[
= r^2 \left[ \frac{3}{2} \theta - 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_0^{2\pi}
\]
\[
= r^2 \left( \frac{3}{2} \cdot 2\pi \right) = 3\pi r^2
\]

**Arc Length**

We already know how to find the length \(L\) of a curve \(C\) given in the form \(y = F(x), a \leq x \leq b\). In fact, if \(F'\) is continuous, then
\[
L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \tag{2}
\]
Suppose that \(C\) can also be described by the parametric equations \(x = f(t)\) and \(y = g(t), \alpha \leq t \leq \beta\), where \(dx/dt = f'(t) > 0\). This means that \(C\) is traversed once, from left to right, as \(t\) increases from \(\alpha\) to \(\beta\) and \(f(\alpha) = a, f(\beta) = b\). Putting Formula 1 into Formula 2 and using the Substitution Rule, we obtain
\[
L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy}{dt}/\frac{dx}{dt}\right)^2} \frac{dx}{dt} \, dt
\]
Since \(dx/dt > 0\), we have
\[
L = \int_\alpha^\beta \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \, dt \tag{3}
\]
Even if $C$ can’t be expressed in the form $y = F(x)$, Formula 3 is still valid but we obtain it by polygonal approximations. We divide the parameter interval $[\alpha, \beta]$ into $n$ subintervals of equal width $\Delta t$. If $t_0, t_1, t_2, \ldots, t_n$ are the endpoints of these subintervals, then $x_i = f(t_i)$ and $y_i = g(t_i)$ are the coordinates of points $P_i(x_i, y_i)$ that lie on $C$ and the polygon with vertices $P_0, P_1, \ldots, P_n$ approximates $C$.

As in Section 7.4, we define the length $L$ of $C$ to be the limit of the lengths of these approximating polygons as $n \to \infty$:

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

The Mean Value Theorem, when applied to $f$ on the interval $[t_{i-1}, t_i]$, gives a number $t_i^*$ in $(t_{i-1}, t_i)$ such that

$$f(t_i) - f(t_{i-1}) = f'(t_i^*)(t_i - t_{i-1})$$

If we let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, this equation becomes

$$\Delta x_i = f'(t_i^*)\Delta t$$

Similarly, when applied to $g$, the Mean Value Theorem gives a number $t_i^{**}$ in $(t_{i-1}, t_i)$ such that

$$\Delta y_i = g'(t_i^{**})\Delta t$$

Therefore

$$|P_{i-1}P_i| = \sqrt{\Delta x_i^2 + \Delta y_i^2} = \sqrt{[f'(t_i^*)\Delta t]^2 + [g'(t_i^{**})\Delta t]^2} = \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\Delta t$$

and so

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\Delta t$$

The sum in (4) resembles a Riemann sum for the function $\sqrt{[f'(t)]^2 + [g'(t)]^2}$ but it is not exactly a Riemann sum because $t_i^* \neq t_i^{**}$ in general. Nevertheless, if $f'$ and $g'$ are continuous, it can be shown that the limit in (4) is the same as if $t_i^*$ and $t_i^{**}$ were equal, namely,

$$L = \int_{\alpha}^{\beta} \sqrt{[f'(t)]^2 + [g'(t)]^2}dt$$

Thus, using Leibniz notation, we have the following result, which has the same form as (3).
THEOREM: If a curve $C$ is described by the parametric equations $x = f(t), y = g(t), \alpha \leq t \leq \beta$, where $f'$ and $g'$ are continuous on $[\alpha, \beta]$ and $C$ is traversed exactly once as $t$ increases from $\alpha$ to $\beta$, then the length of $C$ is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

Notice that the formula in this Theorem is consistent with the general formulas

$$L = \int ds$$

and

$$(ds)^2 = (dx)^2 + (dy)^2$$

of Section 7.4.

EXAMPLE 4: If we use the representation of the unit circle

$$x = \cos t, \quad y = \sin t \quad 0 \leq t \leq 2\pi$$

then $dx/dt = -\sin t$ and $dy/dt = \cos t$, so the above Theorem gives

$$L = \int_{0}^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \int_{0}^{2\pi} \sqrt{\sin^2 t + \cos^2 t} \, dt = \int_{0}^{2\pi} dt = 2\pi$$

as expected. If, on the other hand, we use the representation

$$x = \sin 2t, \quad y = \cos 2t \quad 0 \leq t \leq 2\pi$$

then $dx/dt = 2\cos 2t$, $dy/dt = -2\sin 2t$, and the integral in the above Theorem gives

$$\int_{0}^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \int_{0}^{2\pi} \sqrt{4\cos^2 2t + 4\sin^2 2t} \, dt = \int_{0}^{2\pi} 2dt = 4\pi$$

REMARK: Notice that the integral gives twice the arc length of the circle because as $t$ increases from 0 to $2\pi$, the point $(\sin 2t, \cos 2t)$ traverses the circle twice. In general, when finding the length of a curve $C$ from a parametric representation, we have to be careful to ensure that $C$ is traversed only once as $t$ increases from $\alpha$ to $\beta$.

EXAMPLE 5: Find the length of one arch of the cycloid $x = r(\theta - \sin \theta), \ y = r(1 - \cos \theta)$. 
EXAMPLE 5: Find the length of one arch of the cycloid $x = r(\theta - \sin \theta)$, $y = r(1 - \cos \theta)$.

Solution: From Example 3 we see that one arch is described by the parameter interval $0 \leq \theta \leq 2\pi$. Since

$$\frac{dx}{d\theta} = r(1 - \cos \theta) \quad \text{and} \quad \frac{dy}{d\theta} = r \sin \theta$$

we have

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(1 - \cos \theta)^2 + r^2 \sin^2 \theta} \, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(1 - 2 \cos \theta + \cos^2 \theta + \sin^2 \theta)} \, d\theta$$

$$= r \int_0^{2\pi} \sqrt{2(1 - \cos \theta)} \, d\theta$$

To evaluate this integral we use the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ with $\theta = 2x$, which gives

$$1 - \cos \theta = 2 \sin^2(\theta/2)$$

Since $0 \leq \theta \leq 2\pi$, we have $0 \leq \theta/2 \leq \pi$ and so $\sin(\theta/2) \geq 0$. Therefore

$$\sqrt{2(1 - \cos \theta)} = \sqrt{4 \sin^2(\theta/2)} = 2|\sin(\theta/2)| = 2 \sin(\theta/2)$$

and so

$$L = 2r \int_0^{2\pi} \sin(\theta/2) \, d\theta = 2r [-2 \cos(\theta/2)]_0^{2\pi} = 2r[2 + 2] = 8r$$

EXAMPLE 6: Find the length of the astroid $x = \cos^3 t$, $y = \sin^3 t$, $0 \leq t \leq 2\pi$. 
EXAMPLE 6: Find the length of the astroid \( x = \cos^3 t, \ y = \sin^3 t, \ 0 \leq t \leq 2\pi \).

Solution: Since
\[
\frac{dx}{dt} = 3\cos^2 t(-\sin t) \quad \text{and} \quad \frac{dy}{dt} = 3\sin^2 t\cos t
\]
we have
\[
\int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \int_0^{\pi/2} \sqrt{9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t} \, dt
\]
\[
= \int_0^{\pi/2} \sqrt{9\sin^2 t\cos^2 t(\cos^2 t + \sin^2 t)} \, dt
\]
\[
= \int_0^{\pi/2} \sqrt{9\sin^2 t\cos^2 t} \, dt
\]
\[
= \int_0^{\pi/2} |3\sin t\cos t| \, dt
\]
\[
= \int_0^{\pi/2} 3\sin t\cos t \, dt = \left[ \frac{\sin t}{2} \right]_0^\pi = \frac{3}{2}
\]
It follows that
\[
\text{Length of first-quadrant portion} = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \frac{3}{2}
\]
Therefore the length of the astroid is four times this: \( 4 \cdot \frac{3}{2} = 6 \).