Applications of Taylor Polynomials

Approximating Functions by Polynomials

Suppose that $f(x)$ is equal to the sum of its Taylor series at a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

In Section 8.7 we introduced the notation $T_n(x)$ for the nth partial sum of this series and called it the nth-degree Taylor polynomial of f at a. Thus

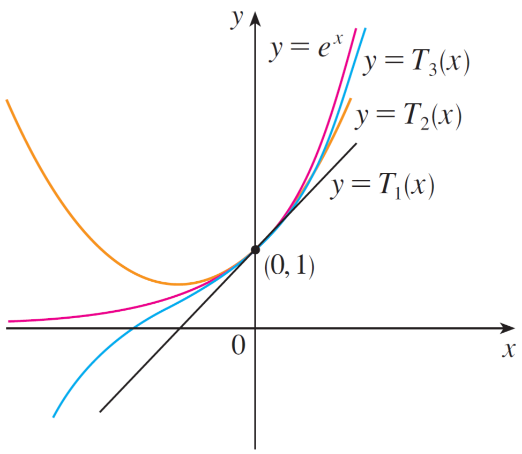
$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Since f is the sum of its Taylor series, we know that $T_n(x) \to f(x)$ as $n \to \infty$ and so T_n can be used as an approximation to f: $f(x) \approx T_n(x)$. Notice that the first-degree Taylor polynomial

$$T_1(x) = f(a) + f'(a)(x-a)$$

is the same as the linearization of f at a that we discussed in Section 2.8. Notice also that T_1 and its derivative have the same values at a that f and f' have. In general, it can be shown that the derivatives of T_n at a agree with those of f up to and including derivatives of order n.

To illustrate these ideas let's take another look at the graphs of $y = e^x$ and its first few Taylor polynomials, as shown in the Figure below. The graph of T_1 is the tangent line to $y = e^x$ at $(0, 1)$; this tangent line is the best linear approximation to e^x near $(0, 1)$. The graph of T_2 is the parabola $y = 1 + x + x^2/2$, and the graph of T_3 is the cubic curve $y = 1 + x + x^2/2 + x^3/6$, which is a closer fit to the exponential curve $y = e^x$ than T_2. The next Taylor polynomial T_4 would be an even better approximation, and so on.



	$x = 0.2$	$x = 3.0$
$T_2(x)$	1.220000	8.500000
$T_4(x)$	1.221400	16.375000
$T_6(x)$	1.221403	19.412500
$T_8(x)$	1.221403	20.009152
$T_{10}(x)$	1.221403	20.079665
e^x	1.221403	20.085537

The values in the Table above give a numerical demonstration of the convergence of the Taylor polynomials $T_n(x)$ to the function $y = e^x$. We see that when $x = 0.2$ the convergence is very rapid, but when $x = 3$ it is somewhat slower. In fact, the farther x is from 0, the more slowly $T_n(x)$ converges to e^x.

When using a Taylor polynomial T_n to approximate a function f, we have to ask the questions: How good an approximation is it? How large should we take n to be in order to achieve a desired accuracy? To answer these questions we need to look at the absolute value of the remainder:

$$|R_n(x)| = |f(x) - T_n(x)|$$

There are three possible methods for estimating the size of the error:

1. If a graphing device is available, we can use it to graph $|R_n(x)|$ and thereby estimate the error.

2. If the series happens to be an alternating series, we can use the Alternating Series Estimation Theorem.

3. In all cases we can use Taylor's Formula, which says that

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1} \quad \text{where} \quad x < z < a$$

EXAMPLE 1:

(a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.

(b) How accurate is this approximation when $7 \le x \le 9$?

Solution:

(a) We have

$$\begin{aligned} f(x) &= \sqrt[3]{x} = x^{1/3} & f(8) &= 2 \\ f'(x) &= \frac{1}{3}x^{-2/3} & f'(8) &= \frac{1}{12} \\ f''(x) &= -\frac{2}{9}x^{-5/3} & f''(8) &= -\frac{1}{144} \\ f'''(x) &= \frac{10}{27}x^{-8/3} \end{aligned}$$

Thus the second-degree Taylor polynomial is

$$T_2(x) = f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2 = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$$

The desired approximation is $\sqrt[3]{x} \approx T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$.

(b) The Taylor series is not alternating when $x < 8$, so we can't use the Alternating Series Estimation Theorem in this example. But using Taylor's Formula we can write

$$R_2(x) = \frac{f'''(z)}{3!}(x-8)^3 = \frac{10}{27}z^{-8/3}\frac{(x-8)^3}{3!} = \frac{5(x-8)^3}{81z^{8/3}}$$

where z lies between 8 and x. In order to estimate the error we note that if $7 \le x \le 9$, then $-1 \le x - 8 \le 1$, so $|x-8| \le 1$ and therefore $|x-8|^3 \le 1$. Also, since $z > 7$, we have

$$z^{8/3} > 7^{8/3} > 179$$

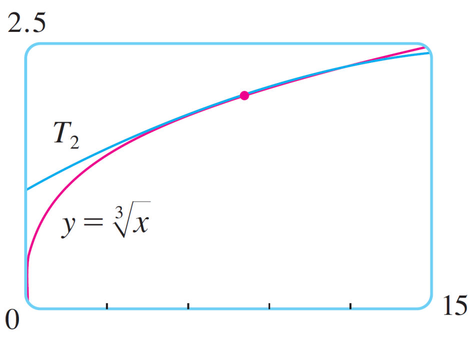
and so

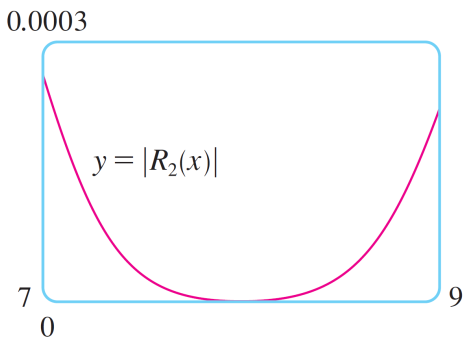
$$|R_2(x)| \le \frac{5|x-8|^3}{81z^{8/3}} < \frac{5 \cdot 1}{81 \cdot 179} < 0.0004$$

Thus if $7 \le x \le 9$, the approximation in part (a) is accurate to within 0.0004.

Let's use a graphing device to check the calculation in Example 1. The Figure below (left) shows that the graphs of $y = \sqrt[3]{x}$ and $y = T_2(x)$ are very close to each other when x is near 8. The Figure on the right shows the graph of $|R_2(x)|$ computed from the expression

$$|R_2(x)| = |\sqrt[3]{x} - T_2(x)|$$





We see from the graph that

$$|R_2(x)| < 0.0003$$

when $7 \le x \le 9$. Thus the error estimate from graphical methods is slightly better than the 0.0004 error estimate from Taylor's Formula in this case.

EXAMPLE 2:

(a) What is the maximum error possible in using the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

when $-0.3 \le x \le 0.3$? Use this approximation to find $\sin 12^{\mathrm{o}}$ correct to six decimal places.

(b) For what values of x is this approximation accurate to within 0.00005?

Solution:

(a) Notice that the Maclaurin series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

is alternating for all nonzero values of x, and the successive terms decrease in size because $|x| < 1$, so we can use the Altemating Series Estimation Theorem. The error in approximating $\sin x$ by the first three terms of its Maclaurin series is at most

$$\left|\frac{x^7}{7!}\right| = \frac{|x|^7}{5040}$$

If $-0.3 \le x \le 0.3$, then $|x| \le 0.3$, so the error is smaller than

$$\frac{(0.3)^7}{5040} \approx 4.3 \times 10^{-8}$$

To find $\sin 12^{\mathrm{o}}$ we first convert to radian measure.

$$\sin 12^{\mathrm{o}} = \sin\left(\frac{12\pi}{180}\right) = \sin\left(\frac{\pi}{15}\right) \approx \frac{\pi}{15} - \left(\frac{\pi}{15}\right)^3 \frac{1}{3!} + \left(\frac{\pi}{15}\right)^5 \frac{1}{5!} \approx 0.20791169$$

Thus, correct to six decimal places, $\sin 12^{\mathrm{o}} \approx 0.207912$.

(b) The error will be smaller than 0.00005 if

$$\frac{|x|^7}{5040} < 0.00005$$

Solving this inequality for x, we get

$$|x|^7 < 0.252 \qquad \text{or} \qquad |x| < (0.252)^{1/7} \approx 0.821$$

So the given approximation is accurate to within 0.00005 when $|x| < 0.82$.

What if we had used Taylor's Formula to solve Example 2? The remainder term is

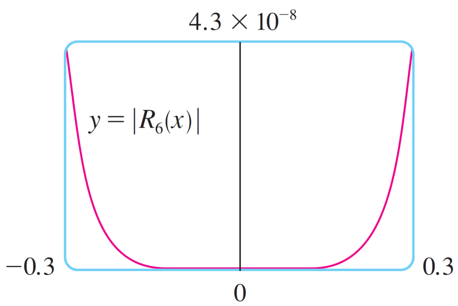
$$R_6(x) = \frac{f^{(7)}(z)}{7!}x^7 = -\cos z\,\frac{x^7}{7!}$$

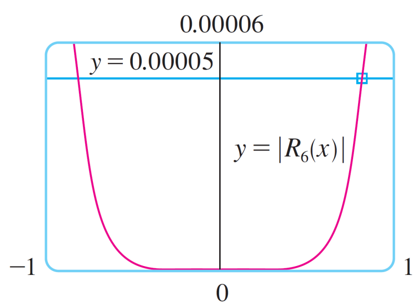
(Note that $T_5 = T_6$.) But $|-\cos z| \leq 1$, so $|R_6(x)| \leq |x|^7/7!$ and we get the same estimates as with the Alternating Series Estimation Theorem.

What about graphical methods? The Figure below (left) shows the graph of

$$|R_6(x)| = \left|\sin x - \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right)\right|$$

and we see from it that $|R_6(x)| < 4.3 \times 10^{-8}$ when $|x| \leq 0.3$. This is the same estimate that we obtained in Example 2. For part (b) we want $|R_6(x)| < 0.00005$, so we graph both $y = |R_6(x)|$ and $y = 0.00005$ in the Figure below (right). By placing the cursor on the right intersection point we find that the inequality is satisfied when $|x| < 0.82$. Again this is the same estimate that we obtained in the solution to Example 2.



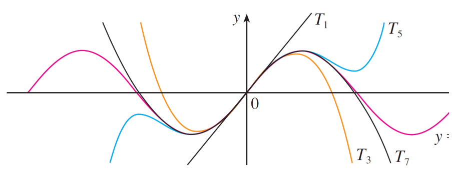


If we had been asked to approximate $\sin 72^{\circ}$ instead of $\sin 12^{\circ}$ in Example 2, it would have been wise to use the Taylor polynomials at $a = \pi/3$ (instead of $a = 0$) because they are better approximations to $\sin x$ for values of x close to $\pi/3$. Notice that 72° is close to 60° (or $\pi/3$ radians) and the derivatives of $\sin x$ are easy to compute at $\pi/3$.

The Figure on the right shows the graphs of the Maclaurin polynomial approximations

$$T_1(x) = x \qquad T_3(x) = x - \frac{x^3}{3!}$$

$$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \qquad T_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$



to the sine curve. You can see that as n increases, $T_n(x)$ is a good approximation to $\sin x$ on a larger and larger interval.