Power Series

DEFINITION: If $c_0, c_1, c_2, \ldots$ are constants and $x$ is a variable, then a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots$$

is called a power series in $x$ and a series of the form

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n = c_0 + c_1 (x - x_0) + c_2 (x - x_0)^2 + \ldots + c_n (x - x_0)^n + \ldots$$

is called a power series in $x - x_0$.

EXAMPLES:

1. $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$

2. $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

3. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

4. $\sum_{n=0}^{\infty} \frac{(x - 1)^n}{n!} = 1 + (x - 1) + \frac{(x - 1)^2}{2!} + \frac{(x - 1)^3}{3!} + \ldots$

5. $\sum_{n=0}^{\infty} (-1)^n \frac{(x + 2)^n}{n!} = 1 - (x + 2) + \frac{(x + 2)^2}{2!} - \frac{(x + 2)^3}{3!} + \ldots$

THEOREM: For any power series $\sum_{n=0}^{\infty} c_n x^n$ exactly one of the following is true:

(a) The series converges only for $x = 0$.

(b) The series converges absolutely (and hence converges) for all real values of $x$.

(c) The series converges absolutely (and hence converges) for all $x$ in some finite open interval $(−R, R)$, and diverges if $x < −R$ or $x > R$. At either of the values $x = −R$ or $x = R$, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.

THEOREM: For a power series $\sum_{n=0}^{\infty} c_n (x - x_0)^n$, exactly one of the following is true:

(a) The series converges only for $x = x_0$.

(b) The series converges absolutely (and hence converges) for all real values of $x$.

(c) The series converges absolutely (and hence converges) for all $x$ in some finite open interval $(x_0 − R, x_0 + R)$, and diverges if $x < x_0 − R$ or $x > x_0 + R$. At either of the values $x = x_0 − R$ or $x = x_0 + R$, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.
EXAMPLE: Find the interval of convergence of the series \( \sum_{n=1}^{\infty} \frac{(x - 5)^n}{n^2} \).

Solution: By the ratio test we have
\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(x - 5)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(x - 5)^n} \right|
\]
\[
= \lim_{n \to \infty} \left[ |x - 5| \left( \frac{n}{n + 1} \right)^2 \right] = |x - 5| \lim_{n \to \infty} \left( \frac{n}{n + 1} \right)^2 = |x - 5|
\]
Thus, the series converges absolutely if
\[|x - 5| < 1 \iff -1 < x - 5 < 1 \iff 4 < x < 6\]
The series diverges if \( x < 4 \) or \( x > 6 \).
Now we examine the endpoints. If \( x = 4 \), then
\[
\sum_{n=1}^{\infty} \frac{(x - 5)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(4 - 5)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}
\]
which converges by the alternating series test. Moreover, it converges absolutely (see below). Similarly, if \( x = 6 \), then
\[
\sum_{n=1}^{\infty} \frac{(x - 5)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(6 - 5)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}
\]
which converges by the \( p \)-series test. So, the interval of convergence is \([4, 6]\) and the radius of convergence is \( R = 1 \).

EXAMPLE: Find the interval of convergence of the series \( \sum_{n=1}^{\infty} \frac{(x + 3)^n}{n} \).
EXAMPLE: Find the interval of convergence of the series \( \sum_{n=1}^{\infty} \frac{(x + 3)^n}{n} \).

Solution: By the ratio test we have

\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(x + 3)^{n+1}}{n + 1} \cdot \frac{n}{(x + 3)^n} \right|
\]

\[
= \lim_{n \to \infty} \left| (x + 3) \cdot \frac{n}{n + 1} \right| = |x + 3| \lim_{n \to \infty} \frac{n}{n + 1} = |x + 3|
\]

Thus, the series converges absolutely if

\[ |x + 3| < 1 \iff -1 < x + 3 < 1 \iff -4 < x < -2 \]

The series diverges if \( x < -4 \) or \( x > -2 \).

Now we examine the endpoints. If \( x = -4 \), then

\[
\sum_{n=1}^{\infty} \frac{(x + 3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-4 + 3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}
\]

which converges by the alternating series test. Moreover, it converges conditionally (see below).
Similarly, if \( x = -2 \), then

\[
\sum_{n=1}^{\infty} \frac{(x + 3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-2 + 3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}
\]

which diverges by the \( p \)-series test. So, the interval of convergence is \([-4, -2]\) and the radius of convergence is \( R = 1 \).

EXAMPLE: Find the interval of convergence of the series \( \sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} (x + 3)^n \).
EXAMPLE: Find the interval of convergence of the series \( \sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} (x + 3)^n \).

Solution: By the ratio test we have

\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1} (n + 1)(x + 3)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{(-1)^n (x + 3)^n} \right| = \lim_{n \to \infty} \frac{(n + 1)(x + 3)}{4n} = \frac{|x + 3|}{4} \lim_{n \to \infty} \frac{n + 1}{n} = \frac{1}{4} |x + 3|.
\]

Thus, the series converges absolutely if

\[
\frac{1}{4} |x + 3| < 1 \iff |x + 3| < 4 \iff -4 < x + 3 < 4 \iff -7 < x < 1.
\]

The series diverges if \( x < -7 \) or \( x > 1 \).

Now we examine the endpoints. If \( x = -7 \), then

\[
\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} (x + 3)^n = \sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} (-4)^n = \sum_{n=1}^{\infty} n \left( \frac{-1(-4)}{4} \right)^n = \sum_{n=1}^{\infty} n.
\]

This series is divergent by the divergence test since \( \lim_{n \to \infty} n = \infty \neq 0 \). Similarly, if \( x = 1 \), then

\[
\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} (x + 3)^n = \sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} 4^n = \sum_{n=1}^{\infty} (-1)^n n.
\]

This series is also divergent by the divergence test, since \( \lim_{n \to \infty} (-1)^n n \) doesn’t exist. So, the interval of convergence is \((-7, 1)\) and the radius of convergence is \( R = 4 \).

EXAMPLE: Find the interval of convergence of the series \( \sum_{n=1}^{\infty} \frac{2^n}{n} (4x - 8)^n \).
EXAMPLE: Find the interval of convergence of the series \( \sum_{n=1}^{\infty} \frac{2^n}{n} (4x - 8)^n \).

Solution: By the ratio test we have

\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{2^{n+1}(4x - 8)^{n+1}}{n+1} \cdot \frac{n}{2^n(4x - 8)^n} \right|
\]

\[
= \lim_{n \to \infty} \left| \frac{2n(4x - 8)}{n+1} \right| = |4x - 8| \lim_{n \to \infty} \frac{2n}{n+1} = 2|4x - 8|
\]

Thus, the series converges absolutely if

\[2|4x - 8| < 1 \iff |4x - 8| < \frac{1}{2} \iff -\frac{1}{2} < 4x - 8 < \frac{1}{2} \iff \frac{15}{2} < 4x < \frac{17}{2} \iff \frac{15}{8} < x < \frac{17}{8}\]

The series diverges if \( x < \frac{15}{8} \) or \( x > \frac{17}{8} \).

Now we examine the endpoints. If \( x = \frac{15}{8} \), then

\[
\sum_{n=1}^{\infty} \frac{2^n}{n} (4x - 8)^n = \sum_{n=1}^{\infty} \frac{2^n}{n} \left( \frac{15}{2} - 8 \right)^n = \sum_{n=1}^{\infty} \frac{2^n}{n} \left( -\frac{1}{2} \right)^n = \sum_{n=1}^{\infty} \frac{2^n(-1)^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}
\]

This series is convergent by the alternating series test. Moreover, it converges conditionally (see below). Similarly, if \( x = \frac{17}{8} \), then

\[
\sum_{n=1}^{\infty} \frac{2^n}{n} (4x - 8)^n = \sum_{n=1}^{\infty} \frac{2^n}{n} \left( \frac{17}{2} - 8 \right)^n = \sum_{n=1}^{\infty} \frac{2^n}{n} \left( \frac{1}{2} \right)^n = \sum_{n=1}^{\infty} \frac{2^n \cdot 1}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{1}{n}
\]

This series is divergent by the \( p \)-test. So, the interval of convergence is \( \left[ \frac{15}{8}, \frac{17}{8} \right) \) and the radius of convergence is \( R = \frac{1}{8} \).

EXAMPLE: Find the interval of convergence of the series \( \sum_{n=0}^{\infty} n!(2x + 1)^n \).
EXAMPLE: Find the interval of convergence of the series \( \sum_{n=0}^{\infty} n!(2x + 1)^n \).

Solution: By the ratio test we have

\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)!(2x + 1)^{n+1}}{n!(2x + 1)^n} \right| \\
= \lim_{n \to \infty} \left| \frac{(n+1)n!(2x + 1)}{n!} \right| \\
= |2x + 1| \lim_{n \to \infty} (n + 1)
\]

Since \( \lim_{n \to \infty} (n + 1) = \infty \), it follows that the series converges if

\[ |2x + 1| = 0 \iff 2x + 1 = 0 \iff x = -\frac{1}{2} \]

and diverges otherwise. The radius of convergence is \( R = 0 \).

EXAMPLE: Find the interval of convergence of the series \( \sum_{n=1}^{\infty} \frac{(x - 6)^n}{n^n} \).
EXAMPLE: Find the interval of convergence of the series \( \sum_{n=1}^{\infty} \frac{(x-6)^n}{n^n} \).

Solution 1: By the root test we have

\[
\rho = \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \sqrt[n]{\frac{(x-6)^n}{n^n}} = \lim_{n \to \infty} \frac{x-6}{n} = |x-6| \lim_{n \to \infty} \frac{1}{n} = |x-6| \cdot 0 = 0
\]

Since \( \rho = 0 < 1 \) regardless of the value of \( x \), this power series will converge absolutely for every \( x \). In these cases we say that the radius of convergence is \( R = \infty \) and interval of convergence the whole number line.

Solution 2: By the ratio test we have

\[
\rho = \lim_{n \to \infty} \frac{|a_{n+1}|}{a_n} = \lim_{n \to \infty} \frac{|(x-6)^{n+1}|}{(n+1)^{n+1} \cdot (x-6)^n} \cdot \frac{n^n}{(n+1)^n} \\
= \lim_{n \to \infty} \frac{|(x-6)^n|}{(n+1)^n} \\
= \lim_{n \to \infty} \frac{|(x-6)^n|}{(n+1)^n \cdot (n+1)} \\
= |x-6| \lim_{n \to \infty} \left[ \left( \frac{n}{n+1} \right)^n \frac{1}{n+1} \right] \\
= |x-6| \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^n \lim_{n \to \infty} \frac{1}{n+1} \\
= |x-6| \cdot e^{-1} \cdot 0 = 0
\]

and the same result follows.

EXAMPLE: Find the interval of convergence of the series \( \sum_{n=1}^{\infty} \frac{x^{2n}}{(-3)^n} \).
EXAMPLE: Find the interval of convergence of the series \( \sum_{n=1}^{\infty} \frac{x^{2n}}{(-3)^n} \).

Solution 1: By the root test we have
\[
\rho = \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \sqrt[n]{\frac{x^{2n}}{(-3)^n}} = \lim_{n \to \infty} \frac{|x^2|}{|-3|} = \frac{x^2}{3}
\]
Thus, the series converges absolutely if
\[
\frac{x^2}{3} < 1 \iff x^2 < 3 \iff -\sqrt{3} < x < \sqrt{3}
\]
The series diverges if \( x < -\sqrt{3} \) or \( x > \sqrt{3} \).

Now we examine the endpoints. If \( x = \pm \sqrt{3} \), then
\[
\sum_{n=1}^{\infty} \frac{x^{2n}}{(-3)^n} = \sum_{n=1}^{\infty} \frac{(\pm \sqrt{3})^{2n}}{(-3)^n} = \sum_{n=1}^{\infty} \frac{(\pm \sqrt{3})^n}{(-3)^n} = \sum_{n=1}^{\infty} \frac{3^n}{(-1)^n \cdot 3^n} = \sum_{n=1}^{\infty} (-1)^n
\]
This series is divergent by the divergence test, since \( \lim_{n \to \infty} (-1)^n \) doesn’t exist. So, the interval of convergence is \((-\sqrt{3}, \sqrt{3})\) and the radius of convergence is \( R = \sqrt{3} \).

Solution 2: By the ratio test we have
\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{2(n+1)}(-3)^n}{x^{2n}(-3)^{n+1}} \right| = \lim_{n \to \infty} \left| \frac{x^{2n+2}}{(-3)^{n+1} \cdot x^{2n}} \right| = \lim_{n \to \infty} \left| \frac{x^2}{-3} \right| = \frac{x^2}{3}
\]
and the same result follows.