

Differential Equations

A **differential equation** is an equation that contains an unknown function and one or more of its derivatives. Here are some examples:

$$y' = 1, \quad y' = x, \quad y' = xy$$

$$y'' + 2y' + y = 0$$

$$\frac{d^3y}{dx^3} + x\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = e^{-x}$$

The **order** of a differential equation is the order of the highest derivative that occurs in the equation.

A function f is called a **solution** of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation. For instance, we know that the general solution of the differential equation $y' = x$ is given by $y = \frac{1}{2}x^2 + C$, where C is an arbitrary constant.

Separable Equations

A **separable equation** is a first-order differential equation that can be written in the form

$$\frac{dy}{dx} = g(x)f(y)$$

The name *separable* comes from the fact that the expression on the right side can be “separated” into a function of x and a function of y . Equivalently, if $f(y) \neq 0$, we could write

$$\frac{dy}{dx} = \frac{g(x)}{h(y)} \tag{1}$$

where $h(y) = 1/f(y)$. To solve this equation we rewrite it in the differential form

$$h(y)dy = g(x)dx$$

so that all y 's are on one side of the equation and all x 's are on the other side. Then we integrate both sides of the equation:

$$\int h(y)dy = \int g(x)dx \tag{2}$$

Equation 2 defines y implicitly as a function of x . In some cases we may be able to solve for y in terms of x .

We use the Chain Rule to justify this procedure: If h and g satisfy (2), then

$$\frac{d}{dx} \left(\int h(y)dy \right) = \frac{d}{dx} \left(\int g(x)dx \right)$$

so

$$\frac{d}{dy} \left(\int h(y)dy \right) \frac{dy}{dx} = g(x)$$

and

$$h(y) \frac{dy}{dx} = g(x)$$

EXAMPLE:

(a) Solve the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2}$.

(b) Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

Solution:

(a) We write the equation in terms of differentials and integrate both sides:

$$y^2 dy = x^2 dx$$

$$\int y^2 dy = \int x^2 dx$$

$$\frac{1}{3}y^3 = \frac{1}{3}x^3 + C$$

hence

$$y = \sqrt[3]{x^3 + 3C}$$

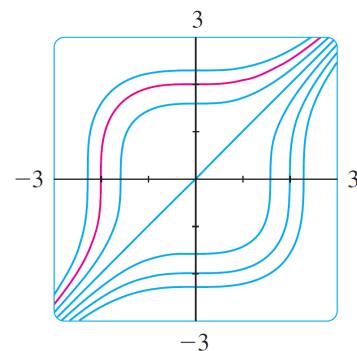
which can be rewritten as

$$y = \sqrt[3]{x^3 + K}$$

where $K = 3C$.

(b) If we put $x = 0$ in the general solution in part (a), we get $y(0) = \sqrt[3]{K}$. To satisfy the initial condition $y(0) = 2$, we must have $\sqrt[3]{K} = 2$ and so $K = 8$. Thus the solution of the initial-value problem is

$$y = \sqrt[3]{x^3 + 8}$$



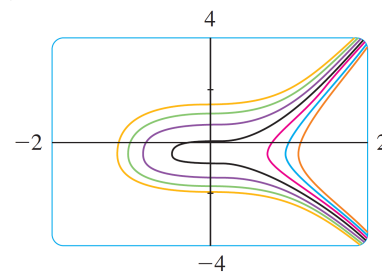
EXAMPLE: Solve the differential equation $\frac{dy}{dx} = \frac{6x^2}{2y + \cos y}$.

Solution: Writing the equation in differential form and integrating both sides, we have

$$(2y + \cos y) dy = 6x^2 dx$$

$$\int (2y + \cos y) dy = \int 6x^2 dx$$

$$y^2 + \sin y = 2x^3 + C$$



where C is a constant. The last equation gives the general solution implicitly. In this case it's impossible to solve the equation to express y explicitly as a function of x .

EXAMPLE: Solve the equation $y' = x^2 y$.

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Solution: First we rewrite the equation using Leibniz notation:

$$\frac{dy}{dx} = x^2y$$

If $y \neq 0$, we can rewrite it in differential notation and integrate:

$$\begin{aligned} \frac{dy}{y} &= x^2 dx & y \neq 0 \\ \int \frac{dy}{y} &= \int x^2 dx \\ \ln |y| &= \frac{x^3}{3} + C \end{aligned}$$

This equation defines y implicitly as a function of x . But in this case we can solve explicitly for y as follows:

$$|y| = e^{\ln |y|} = e^{(x^3/3)+C} = e^C e^{x^3/3}$$

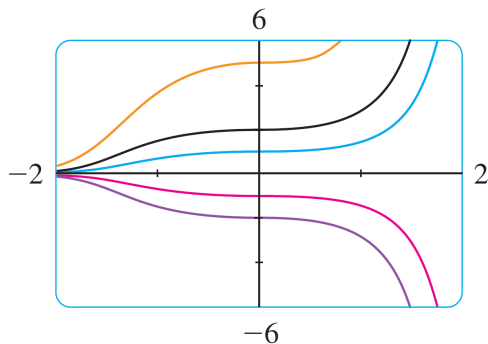
so

$$y = \pm e^C e^{x^3/3}$$

We can easily verify that the function $y = 0$ is also a solution of the given differential equation. So we can write the general solution in the form

$$y = Ae^{x^3/3}$$

where A is an arbitrary constant ($A = e^C$, or $A = -e^C$, or $A = 0$).



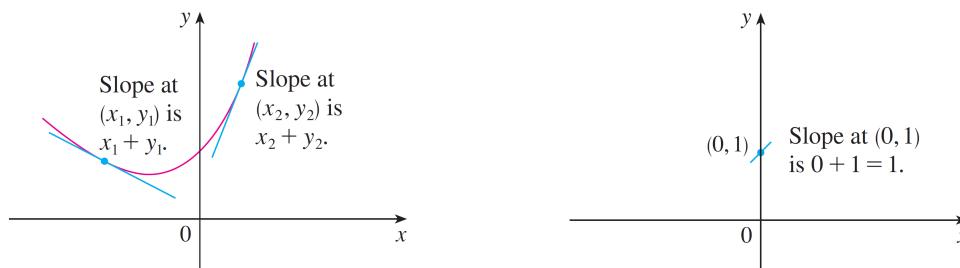
Direction Fields

Unfortunately, it's impossible to solve most differential equations in the sense of obtaining an explicit formula for the solution. In this section we show that, despite the absence of an explicit solution, we can still learn a lot about the solution through a graphical approach (direction fields).

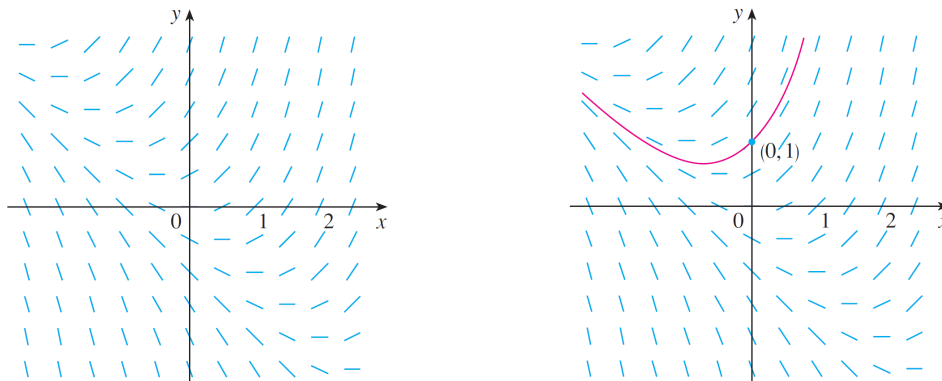
Suppose we are asked to sketch the graph of the solution of the initial-value problem

$$y' = x + y \qquad y(0) = 1$$

We don't know a formula for the solution, so how can we possibly sketch its graph? Let's think about what the differential equation means. The equation $y' = x + y$ tells us that the slope at any point (x, y) on the graph (called the *solution curve*) is equal to the sum of the x - and y -coordinates of the point (see Figure 1 below). In particular, because the curve passes through the point $(0, 1)$, its slope there must be $0 + 1 = 1$. So a small portion of the solution curve near the point $(0, 1)$ looks like a short line segment through $(0, 1)$ with slope 1. (See Figure 2 below.)



As a guide to sketching the rest of the curve, let's draw short line segments at a number of points (x, y) with slope $x + y$. The result is called a *direction field* and is shown in Figure 1 below. For instance, the line segment at the point $(1, 2)$ has slope $1 + 2 = 3$. The direction field allows us to visualize the general shape of the solution curves by indicating the direction in which the curves proceed at each point.



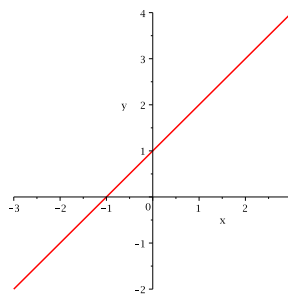
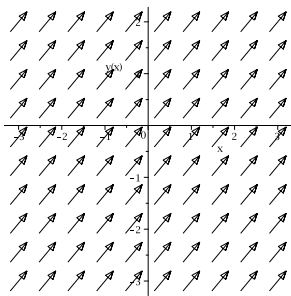
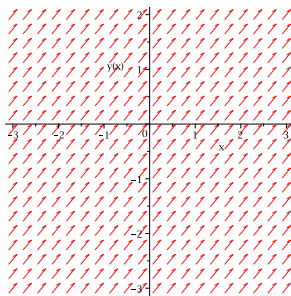
Now we can sketch the solution curve through the point $(0, 1)$ by following the direction field as in Figure 2 above. Notice that we have drawn the curve so that it is parallel to nearby line segments.

In general, suppose we have a first-order differential equation of the form

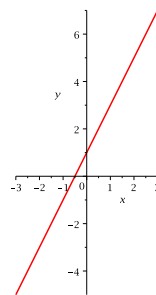
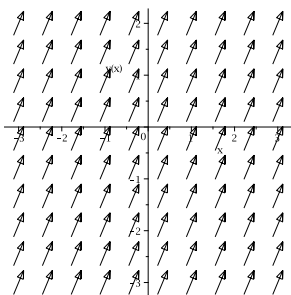
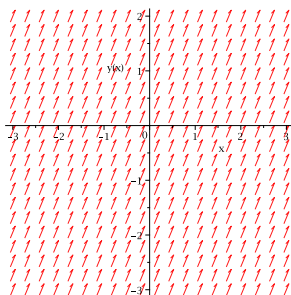
$$y' = F(x, y)$$

where $F(x, y)$ is some expression in x and y . The differential equation says that the slope of a solution curve at a point (x, y) on the curve is $F(x, y)$. If we draw short line segments with slope $F(x, y)$ at several points (x, y) , the result is called a **direction field** (or **slope field**). These line segments indicate the direction in which a solution curve is heading, so the direction field helps us visualize the general shape of these curves.

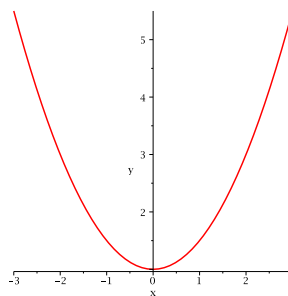
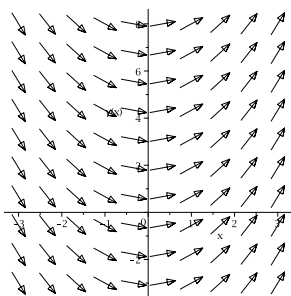
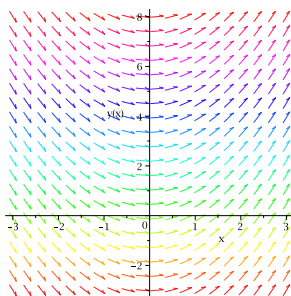
EXAMPLES:



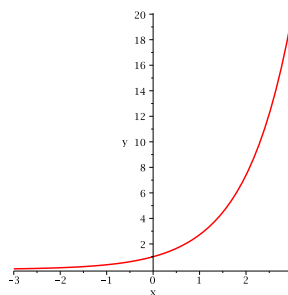
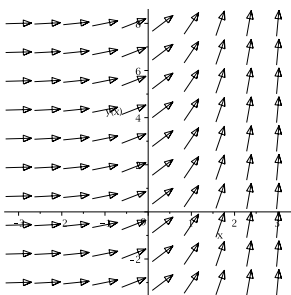
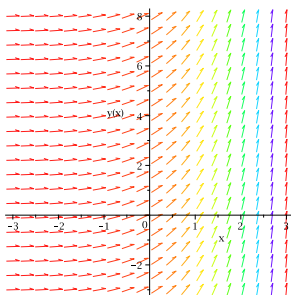
$$y' = 1$$



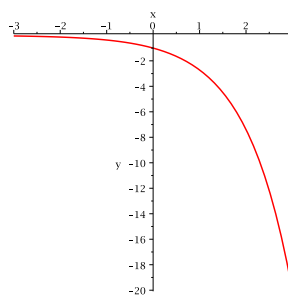
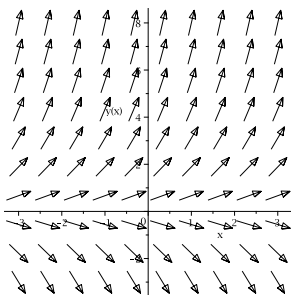
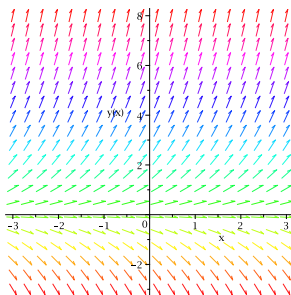
$$y' = 2$$



$$y' = x$$

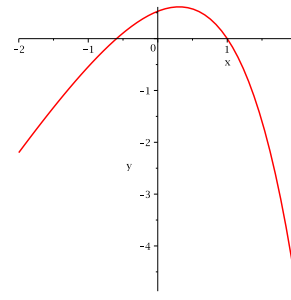
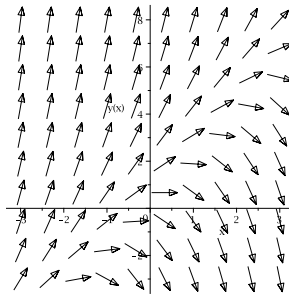
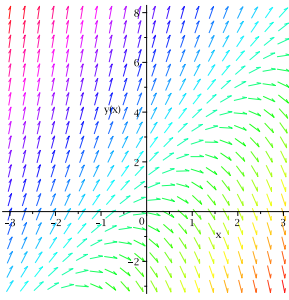


$$y' = e^x$$

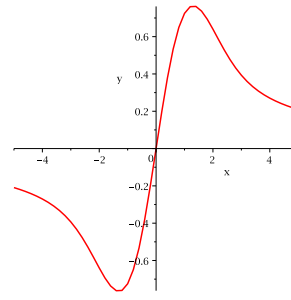
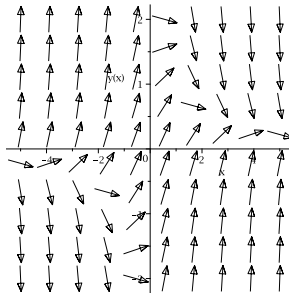
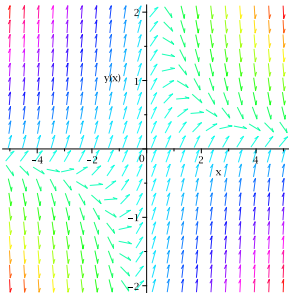


$$y' = y$$

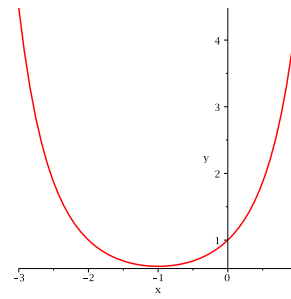
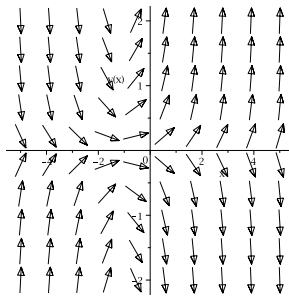
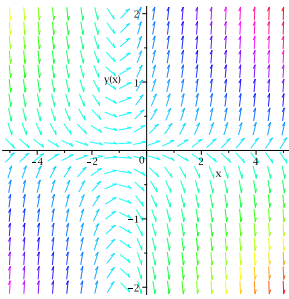
EXAMPLES:



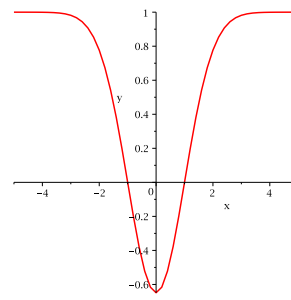
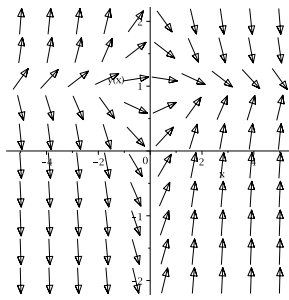
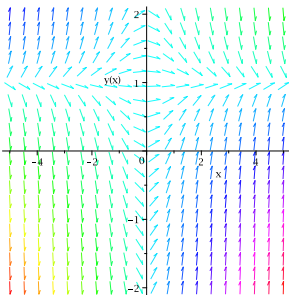
$$y' = y - 2x$$



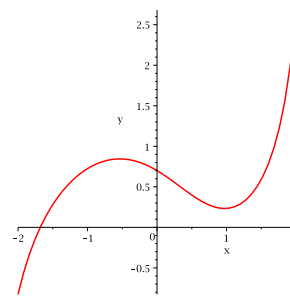
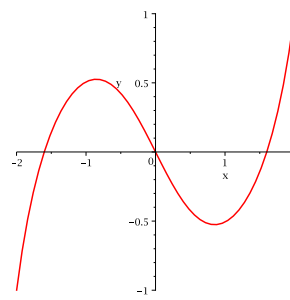
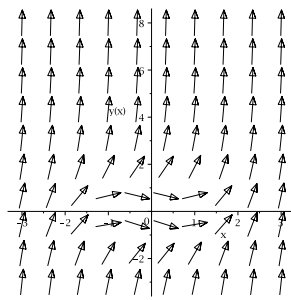
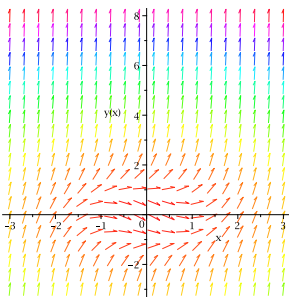
$$y' = 1 - yx$$



$$y' = y + yx$$



$$y' = x - xy$$



$$y' = x^2 + y^2 - 1$$

EXAMPLE:

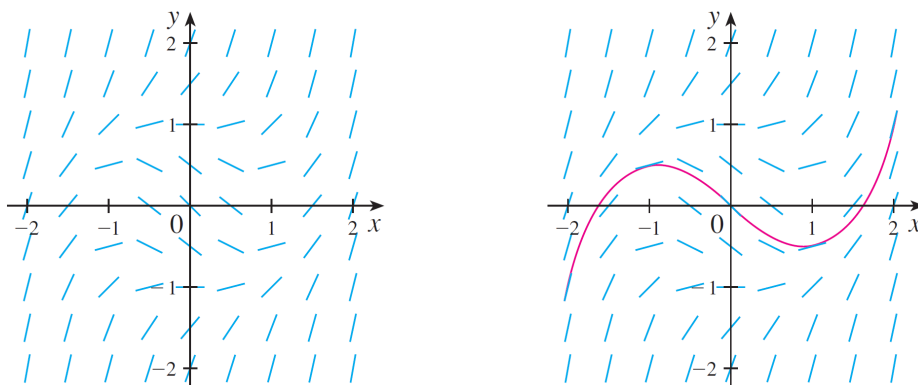
- (a) Sketch the direction field for the differential equation $y' = x^2 + y^2 - 1$.
- (b) Use part (a) to sketch the solution curve that passes through the origin.

Solution:

- (a) We start by computing the slope at several points in the following chart:

x	-2	-1	0	1	2	-2	-1	0	1	2
y	0	0	0	0	0	1	1	1	1	1
$y' = x^2 + y^2 - 1$	3	0	-1	0	3	4	1	0	1	4

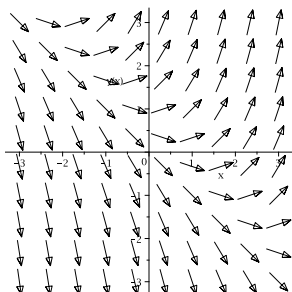
Now we draw short line segments with these slopes at these points. The result is the direction field shown in the Figure below (left).



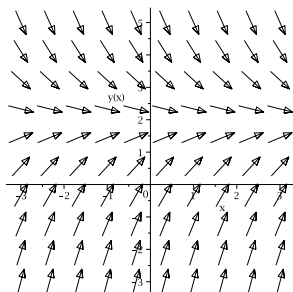
- (b) We start at the origin and move to the right in the direction of the line segment (which has slope -1). We continue to draw the solution curve so that it moves parallel to the nearby line segments. The resulting solution curve is shown in the Figure above (right). Returning to the origin, we draw the solution curve to the left as well.

EXAMPLE: Match the differential equation with its direction field

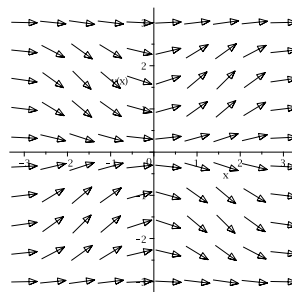
- 1. $y' = 2 - y$
- 2. $y' = x(2 - y)$
- 3. $y' = x + y - 1$
- 4. $y' = \sin x \sin y$



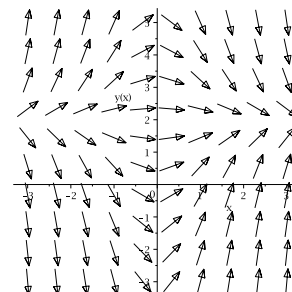
(I)



(II)



(III)



(IV)