Applications to Physics and Engineering

Work

The term *work* is used in everyday language to mean the total amount of effort required to perform a task. In physics it has a technical meaning that depends on the idea of a *force*. Intuitively, you can think of a force as describing a push or pull on an object — for example, a horizontal push of a book across a table or the downward pull of the Earth's gravity on a ball. In general, if an object moves along a straight line with position function $s(t)$, then the **force** F on the object (in the same direction) is defined by Newton's Second Law of Motion as the product of its mass m and its acceleration:

$$F = m\frac{d^2s}{dt^2} \tag{1}$$

In the SI metric system, the mass is measured in kilograms (kg), the displacement in meters (m), the time in seconds (s), and the force in newtons (N = kg $\cdot$ m/s^2). Thus a force of 1 N acting on a mass of 1 kg produces an acceleration of 1 m/s^2. In the US Customary system the fundamental unit is chosen to be the unit of force, which is the pound.

In the case of constant acceleration, the force F is also constant and the work done is defined to be the product of the force F and the distance d that the object moves:

$$W = Fd \qquad \text{work} = \text{force} \times \text{distance} \tag{2}$$

If F is measured in newtons and d in meters, then the unit for W is a newton-meter, which is called a joule (J). If F is measured in pounds and d in feet, then the unit for W is a foot-pound (ft-lb), which is about 1.36 J.

EXAMPLE:

(a) How much work is done in lifting a 1.2-kg book off the floor to put it on a desk that is 0.7 m high? Use the fact that the acceleration due to gravity is $g = 9.8$ m/s^2.

(b) How much work is done in lifting a 20-lb weight 6 ft off the ground?

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(b) How much work is done in lifting a 20-lb weight 6 ft off the ground?

Solution:

(a) The force exerted is equal and opposite to that exerted by gravity, so Equation 1 gives

$$F = mg = (1.2)(9.8) = 11.76 \text{ N}$$

and then Equation 2 gives the work done as

$$W = Fd = (11.76)(0.7) \approx 8.2 \text{ J}$$

(b) Here the force is given as $F = 20$ lb, so the work done is

$$W = Fd = 20 \cdot 6 = 120 \text{ ft-lb}$$

Notice that in part (b), unlike part (a), we did not have to multiply by g because we were given the *weight* (which is a force) and not the mass of the object.

Equation 2 defines work as long as the force is constant, but what happens if the force is variable? Let's suppose that the object moves along the x-axis in the positive direction, from $x = a$ to $x = b$, and at each point x between a and b a force $f(x)$ acts on the object, where f is a continuous function. We divide the interval $[a, b]$ into n subintervals with end points $x_0, x_1, \ldots, x_n$ and equal width Δx. We choose a sample point x_i^* in the ith subinterval $[x_{i-1}, x_i]$. Then the force at that point is $f(x_i^*)$. If n is large, then Δx is small, and since f is continuous, the values of f don't change very much over the interval $[x_{i-1}, x_i]$. In other words, f is almost constant on the interval and so the work W_i that is done in moving the particle from x_{i-1} to x_i is approximately given by Equation 2:

$$W_i \approx f(x_i^*)\Delta x$$

Thus we can approximate the total work by

$$W \approx \sum_{i=1}^{n} f(x_i^*)\Delta x \tag{3}$$

It seems that this approximation becomes better as we make n larger. Therefore, we define the **work done in moving the object from a to b** as the limit of this quantity as $n \to \infty$. Since the right side of (3) is a Riemann sum, we recognize its limit as being a definite integral and so

$$\boxed{W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)dx} \tag{4}$$

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Solution: We have

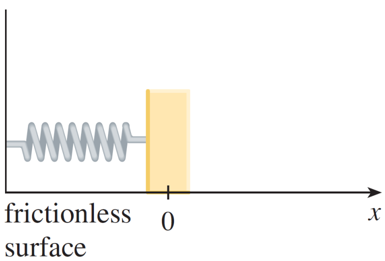
$$W = \int_1^3 (x^2 + 2x)dx = \left[\frac{x^3}{3} + x^2\right]_1^3 = \frac{50}{3}$$

The work done is $16\frac{2}{3}$ ft-lb.

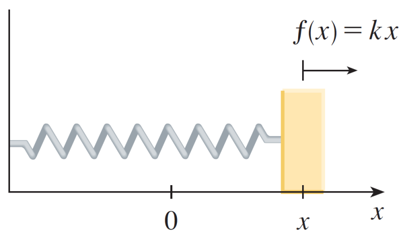
In the next example we use a law from physics: **Hooke's Law** states that the force required to maintain a spring stretched x units beyond its natural length is proportional to x:

$$\boxed{f(x) = kx}$$

where k is a positive constant (called the **spring constant**). Hooke's Law holds provided that x is not too large.



(a) Natural position of spring



(b) Stretched position of spring

EXAMPLE: A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done in stretching the spring from 15 cm to 18 cm?

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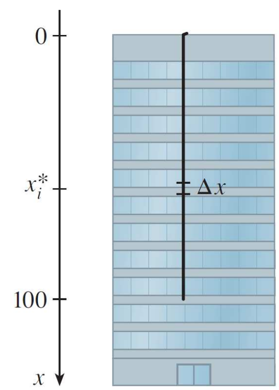
Solution: According to Hooke's Law, the force required to hold the spring stretched x meters beyond its natural length is $f(x) = kx$. When the spring is stretched from 10 cm to 15 cm, the amount stretched is 5 cm = 0.05 m. This means that $f(0.05) = 40$, so

$$0.05k = 40 \quad \Longrightarrow \quad k = \frac{40}{0.05} = 800$$

Thus $f(x) = 800x$ and the work done in stretching the spring from 15 cm to 18 cm is

$$W = \int_{0.05}^{0.08} 800x dx = 800 \frac{x^2}{2}\bigg]_{0.05}^{0.08} = 400\left[(0.08)^2 - (0.05)^2\right] = 1.56 \text{ J}$$

EXAMPLE: A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?



Solution: Here we don't have a formula for the force function, but we can use an argument similar to the one that led to Definition 4.

Let's place the origin at the top of the building and the x-axis pointing downward as in the figure above. We divide the cable into small parts with length Δx. If x_i^* is a point in the ith such interval, then all points in the interval are lifted by approximately the same amount, namely x_i^*. The cable weighs 2 pounds per foot, so the weight of the ith part is $2\Delta x$. Thus the work done on the ith part, in foot-pounds, is

$$\underbrace{(2\Delta x)}_{\text{force}} \cdot \underbrace{x_i^*}_{\text{distance}} = 2x_i^*\Delta x$$

We get the total work done by adding all these approximations and letting the number of parts become large (so $\Delta x \to 0$):

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 2x_i^*\Delta x = \int_0^{100} 2x dx = x^2\Big]_0^{100} = 10,000 \text{ ft-lb}$$

EXAMPLE: A tank has the shape of an inverted circular cone with height 10 m and base radius 4 m. It is filled with water to a height of 8 m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000 kg/m^3.)

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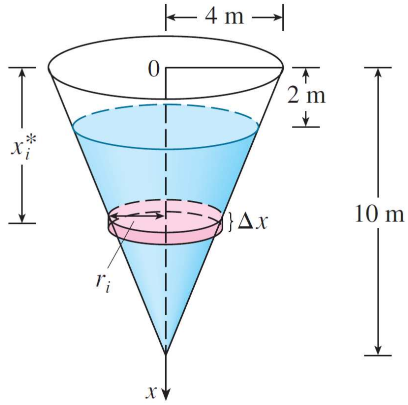
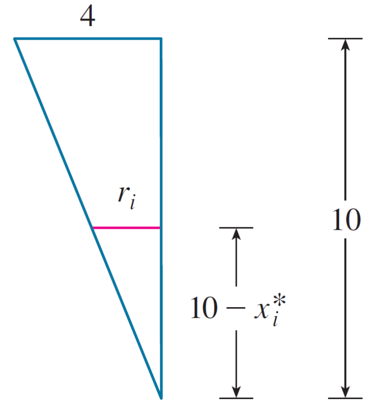




Solution: Let's measure depths from the top of the tank by introducing a vertical coordinate line. The water extends from a depth of 2 m to a depth of 10 m and so we divide the interval $[2, 10]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and choose x_i^* in the ith subinterval. This divides the water into n layers. The ith layer is approximated by a circular cylinder with radius r_i and height Δx. We can compute r_i from similar triangles as follows:

$$\frac{r_i}{10 - x_i^*} = \frac{4}{10} \quad \Longrightarrow \quad r_i = \frac{2}{5}\left(10 - x_i^*\right)$$

Thus an approximation to the volume of the ith layer of water is

$$V_i \approx \pi r_i^2 \Delta x = \frac{4\pi}{25}\left(10 - x_i^*\right)^2 \Delta x$$

and so its mass is

$$m_i = \text{density} \times \text{volume} \approx 1000 \cdot \frac{4\pi}{25}\left(10 - x_i^*\right)^2 \Delta x = 160\pi \left(10 - x_i^*\right)^2 \Delta x$$

The force required to raise this layer must overcome the force of gravity and so

$$F_i = m_i g \approx 9.8 \cdot 160\pi \left(10 - x_i^*\right)^2 \Delta x \approx 1568\pi \left(10 - x_i^*\right)^2 \Delta x$$

Each particle in the layer must travel a distance of approximately x_i^*. The work W_i done to raise this layer to the top is approximately the product of the force F_i and the distance x_i^*:

$$W_i \approx F_i x_i^* \approx 1568\pi x_i^* \left(10 - x_i^*\right)^2 \Delta x$$

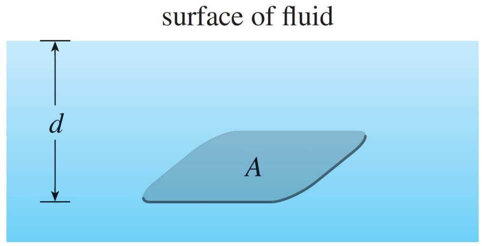
To find the total work done in emptying the entire tank, we add the contributions of each of the n layers and then take the limit as $n \to \infty$:

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} 1568\pi x_i^* \left(10 - x_i^*\right)^2 \Delta x = \int_2^{10} 1568\pi x \left(10 - x\right)^2 dx \\ &= 1568\pi \int_2^{10} \left(100x - 20x^2 + x^3\right) dx = 1568\pi \left[50x^2 - \frac{20x^3}{3} + \frac{x^4}{4}\right]_2^{10} \\ &= 1568\pi \cdot \frac{2048}{3} \approx 3.4 \times 10^6 \text{ J} \end{aligned}$$

Hydrostatic Pressure and Force

Deep-sea divers realize that water pressure increases as they dive deeper. This is because the weight of the water above them increases.

In general, suppose that a thin horizontal plate with area A square meters is submerged in a fluid of density ρ kilograms per cubic meter at a depth d meters below the surface of the fluid.



The fluid directly above the plate has volume $V = Ad$, so its mass is $m = \rho V = \rho Ad$. The force exerted by the fluid on the plate is therefore

$$F = mg = \rho g A d$$

where g is the acceleration due to gravity. The pressure P on the plate is defined to be the force per unit area:

$$P = \frac{F}{A} = \rho g d$$

The SI unit for measuring pressure is newtons per square meter, which is called a pascal (abbreviation: 1 $\mathrm{N/m^2}$ = 1 Pa). Since this is a small unit, the kilopascal (kPa) is often used. For instance, because the density of water is $\rho = 1000\ \mathrm{kg/m^3}$, the pressure at the bottom of a swimming pool 2 m deep is

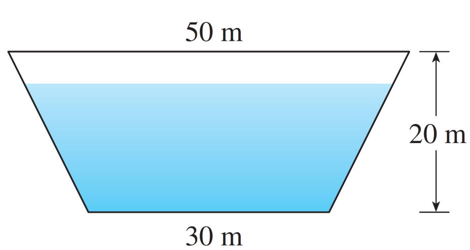
$$P = \rho g d = 1000\ \mathrm{kg/m^3} \times 9.8\ \mathrm{m/s^2} \times 2\ \mathrm{m} = 19,600\ \mathrm{Pa}\ = 19.6\ \mathrm{kPa}$$

An important principle of fluid pressure is the experimentally verified fact that *at any point in a liquid the pressure is the same in all directions.* (A diver feels the same pressure on nose and both ears.) Thus, the pressure in *any* direction at a depth d in a fluid with mass density ρ is given by

$$\boxed{P = \rho g d = \delta d} \tag{5}$$

This helps us determine the hydrostatic force against a *vertical* plate or wall or dam in a fluid. This is not a straightforward problem because the pressure is not constant but increases as the depth increases.

EXAMPLE: A dam has the shape of the trapezoid shown in the figure below. The height is 20 m and the width is 50 m at the top and 30 m at the bottom. Find the force on the dam due to hydrostatic pressure if the water level is 4 m from the top of the dam.



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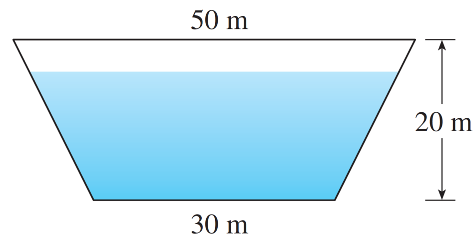
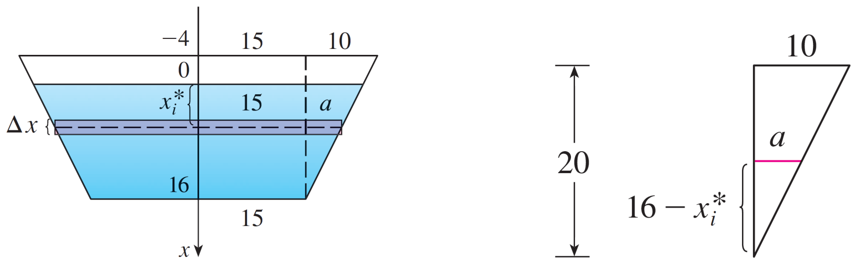


Solution: We choose a vertical x-axis with origin at the surface of the water:



The depth of the water is 16 m, so we divide the interval $[0, 16]$ into subintervals of equal length with endpoints x_i and we choose $x_i^* \in [x_{i-1}, x_i]$. The ith horizontal strip of the dam is approximated by a rectangle with height Δx and width w_i, where, from similar triangles,

$$\frac{a}{16 - x_i^*} = \frac{10}{20} \quad \Longrightarrow \quad a = \frac{16 - x_i^*}{2} = 8 - \frac{1}{2}x_i^*$$

and so

$$w_i = 2(15 + a) = 2\left(15 + 8 - \frac{1}{2}x_i^*\right) = 46 - x_i^*$$

If A_i is the area of the ith strip, then

$$A_i \approx w_i \Delta x = (46 - x_i^*)\,\Delta x$$

If Δx is small, then the pressure P_i on the ith strip is almost constant and we can use Equation 5 to write

$$P_i \approx 1000 g x_i^*$$

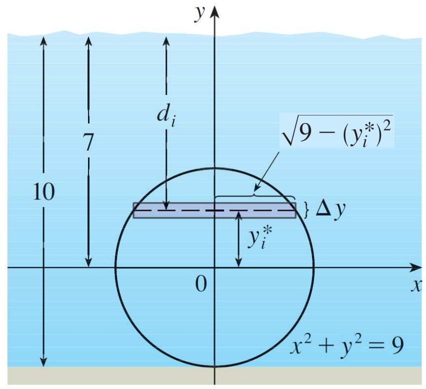
The hydrostatic force F_i acting on the ith strip is the product of the pressure and the area:

$$F_i = P_i A_i \approx 1000 g x_i^* (46 - x_i^*) \Delta x$$

Adding these forces and taking the limit as $n \to \infty$, we obtain the total hydrostatic force on the dam:

$$\begin{aligned} F &= \lim_{n\to\infty} \sum_{i=1}^{n} 1000 g x_i^* (46 - x_i^*) \Delta x = \int_0^{16} 1000 g x (46 - x) dx \\ &= 1000 \cdot 9.8 \int_0^{16} \left(46x - x^2\right) dx = 9800 \left[23x^2 - \frac{x^3}{3}\right]_0^{16} \\ &\approx 4.43 \times 10^7 \text{ N} \end{aligned}$$

EXAMPLE: Find the hydrostatic force on one end of a cylindrical drum with radius 3 ft if the drum is submerged in water 10 ft deep.



Solution: In this example it is convenient to choose the axes as in figure above so that the origin is placed at the center of the drum. Then the circle has a simple equation, $x^2+y^2=9$. We divide the circular region into horizontal strips of equal width. From the equation of the circle, we see that the length of the ith strip is $2\sqrt{9-(y_i^*)^2}$ and so its area is

$$A_i = 2\sqrt{9-(y_i^*)^2}\Delta y$$

The pressure on this strip is approximately

$$\delta d_i = 62.5\,(7-y_i^*)$$

and so the force on the strip is approximately

$$\delta d_i A_i = 62.5\,(7-y_i^*)\,2\sqrt{9-(y_i^*)^2}\Delta y$$

The total force is obtained by adding the forces on all the strips and taking the limit:

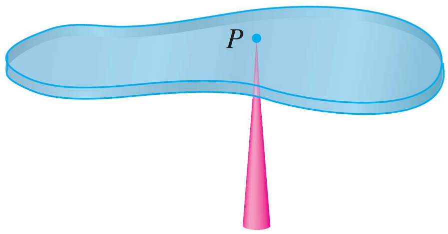
$$\begin{aligned} F &= \lim_{n\to\infty}\sum_{i=1}^{n} 62.5\,(7-y_i^*)\,2\sqrt{9-(y_i^*)^2}\Delta y \\ &= 125\int_{-3}^{3}(7-y)\sqrt{9-y^2}dy \\ &= 125\cdot 7\int_{-3}^{3}\sqrt{9-y^2}dy - 125\int_{-3}^{3} y\sqrt{9-y^2}dy \end{aligned}$$

The second integral is 0 because the integrand is an odd function. The first integral can be evaluated using the trigonometric substitution $y=3\sin\theta$, but it's simpler to observe that it is the area of a semicircular disk with radius 3. Thus

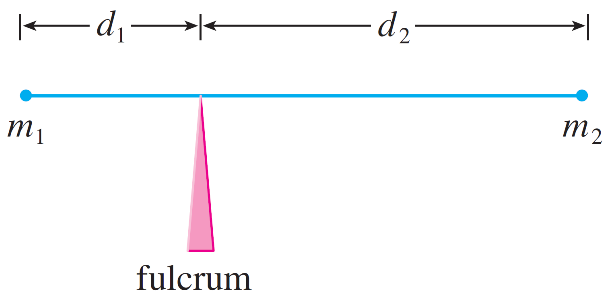
$$F = 875\int_{-3}^{3}\sqrt{9-y^2}dy = 875\cdot\frac{1}{2}\pi\cdot 3^2 = \frac{7875\pi}{2} \approx 12,370 \text{ lb}$$

Moments and Centers of Mass

Our main objective here is to find the point P on which a thin plate of any given shape balances horizontally. This point is called the **center of mass** (or center of gravity) of the plate.



We first consider the simpler situation illustrated in the figure below, where two masses m_1 and m_2 are attached to a rod of negligible mass on opposite sides of a fulcrum and at distances d_1 and d_2 from the fulcrum.



The rod will balance if

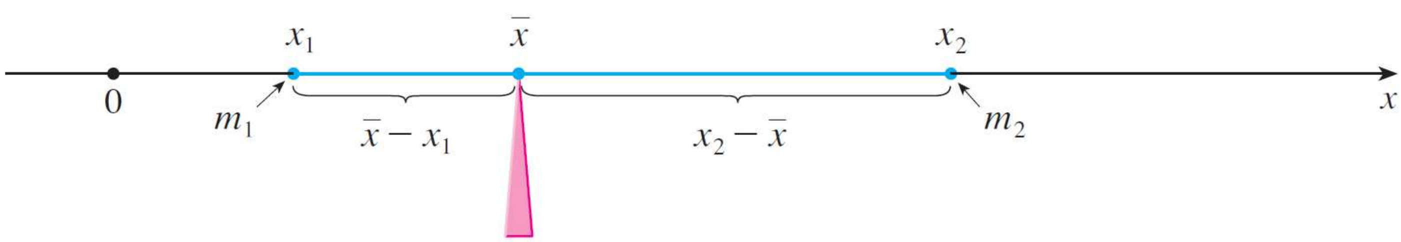
$$\boxed{m_1 d_1 = m_2 d_2} \tag{6}$$

This is an experimental fact discovered by Archimedes and called the Law of the Lever.

Now suppose that the rod lies along the x-axis with m_1 at x_1 and m_2 at x_2 and the center of mass at $\overline{x}$. If we compare the figure above and the figure below, we see that $d_1 = \overline{x} - x_1$ and $d_2 = x_2 - \overline{x}$ and so Equation 6 gives

$$m_1(\overline{x} - x_1) = m_2(x_2 - \overline{x}) \quad \Longrightarrow \quad m_1\overline{x} + m_2\overline{x} = m_1x_1 + m_2x_2 \quad \Longrightarrow \quad \boxed{\overline{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}} \tag{7}$$

The numbers m_1x_1 and m_2x_2 are called the **moments** of the masses m_1 and m_2 (with respect to the origin), and Equation 7 says that the center of mass $\overline{x}$ is obtained by adding the moments of the masses and dividing by the total mass $m = m_1 + m_2$.

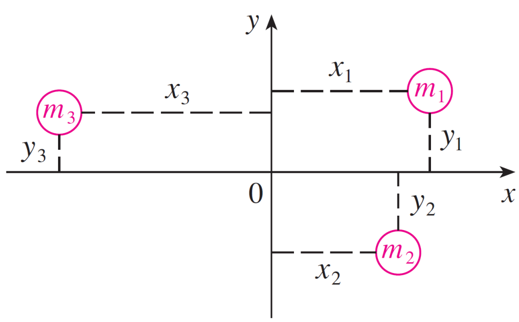


In general, if we have a system of n particles with masses $m_1, m_2, \ldots, m_n$ located at the points $x_1, x_2, \ldots, x_n$ on the x-axis, it can be shown similarly that the center of mass of the system is located at

$$\boxed{\overline{x} = \frac{\displaystyle\sum_{i=1}^{n} m_i x_i}{\displaystyle\sum_{i=1}^{n} m_i} = \frac{\displaystyle\sum_{i=1}^{n} m_i x_i}{m}} \tag{8}$$

where $m = \sum_{i=1}^{n} m_i$ is the total mass of the system, and the sum of the individual moments $M = \sum_{i=1}^{n} m_i x_i$ is called the **moment of the system about the origin**. Then Equation 8 could be rewritten as $m\overline{x} = M$, which says that if the total mass were considered as being concentrated at the center of mass, then its moment would be the same as the moment of the system.

Now we consider a system of n particles with masses $m_1, m_2, \ldots, m_n$ located at the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in the xy-plane:



By analogy with the one-dimensional case, we define the **moment of the system about the y-axis** to be

$$M_y = \sum_{i=1}^{n} m_i x_i \tag{9}$$

and the **moment of the system about the x-axis** as

$$M_x = \sum_{i=1}^{n} m_i y_i \tag{10}$$

Then M_y measures the tendency of the system to rotate about the y-axis and M_x measures the tendency to rotate about the x-axis. As in the one-dimensional case, the coordinates $(\overline{x}, \overline{y})$ of the center of mass are given in terms of the moments by the formulas

$$\boxed{\overline{x} = \frac{M_y}{m}} \quad \text{and} \quad \boxed{\overline{y} = \frac{M_x}{m}} \tag{11}$$

where $m = \sum_{i=1}^{n} m_i$ is the total mass. Since $m\overline{x} = M_y$ and $m\overline{y} = M_x$, the center of mass $(\overline{x}, \overline{y})$ is the point where a single particle of mass m would have the same moments as the system.

EXAMPLE: Find the moments and center of mass of the system of objects that have masses 3, 4, and 8 at the points $(-1, 1), (2, -1)$, and $(3, 2)$.

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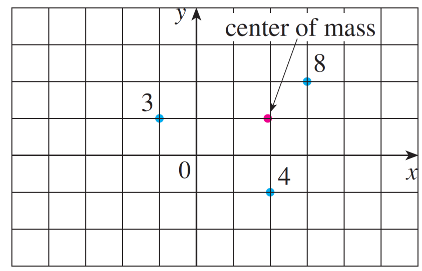


Solution: We use Equations 9 and 10 to compute the moments:

$$M_y = 3 \cdot (-1) + 4 \cdot 2 + 8 \cdot 3 = 29$$

$$M_x = 3 \cdot 1 + 4 \cdot (-1) + 8 \cdot 2 = 15$$

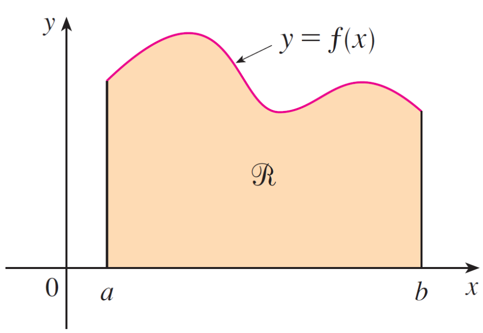
Since $m = 3 + 4 + 8 = 15$, we use Equations 11 to obtain

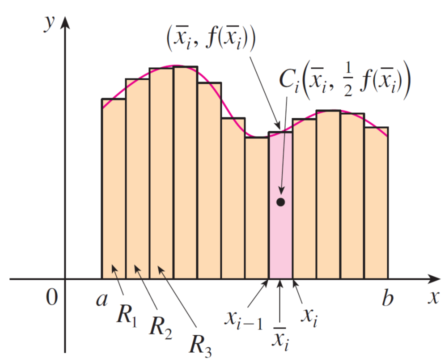
$$\overline{x} = \frac{M_y}{m} = \frac{29}{15} \quad \text{and} \quad \overline{y} = \frac{M_x}{m} = \frac{15}{15} = 1$$

Thus the center of mass is $\left(1\frac{14}{15}, 1\right)$.

Next we consider a flat plate (called a *lamina*) with uniform density ρ that occupies a region $\mathcal{R}$ of the plane. We wish to locate the center of mass of the plate, which is called the **centroid** of $\mathcal{R}$. In doing so we use the following physical principles: The **symmetry principle** says that if $\mathcal{R}$ is symmetric about a line l, then the centroid of $\mathcal{R}$ lies on l. (If $\mathcal{R}$ is reflected about l, then $\mathcal{R}$ remains the same so its centroid remains fixed. But the only fixed points lie on l.) Thus the centroid of a rectangle is its center. Moments should be defined so that if the entire mass of a region is concentrated at the center of mass, then its moments remain unchanged. Also, the moment of the union of two nonoverlapping regions should be the sum of the moments of the individual regions.

Suppose that the region $\mathcal{R}$ is of the type shown in first figure below; that is, $\mathcal{R}$ lies between the lines $x = a$ and $x = b$, above the x-axis, and beneath the graph of f, where f is a continuous function.





We divide the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$, and equal width Δx. We choose the sample point x_i^* to be the midpoint $\overline{x}_i$ of the ith subinterval, that is, $\overline{x}_i = (x_{i-1} + x_i)/2$. This determines the polygonal approximation to $\mathcal{R}$ shown in the second figure above. The centroid of the ith approximating rectangle R_i is its center $C_i\left(\overline{x}_i, \frac{1}{2} f(\overline{x}_i)\right)$. Its area is $f(\overline{x}_i)\Delta x$, so its mass is

$$\rho f(\overline{x}_i)\Delta x$$

The moment of R_i about the y-axis is the product of its mass and the distance from C_i, to the y-axis, which is $\overline{x}_i$. Thus

$$M_y(R_i) = [\rho f(\overline{x}_i)\Delta x]\overline{x}_i = \rho \overline{x}_i f(\overline{x}_i)\Delta x$$

Adding these moments, we obtain the moment of the polygonal approximation to $\mathcal{R}$, and then by taking the limit as $n \to \infty$ we obtain the moment of $\mathcal{R}$ itself about the y-axis:

$$\boxed{M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \overline{x}_i f(\overline{x}_i)\Delta x = \rho \int_a^b x f(x)dx}$$

In a similar fashion we compute the moment of R_i about the x-axis as the product of its mass and the distance from C_i to the x-axis:

$$M_x(R_i) = [\rho f(\overline{x}_i)\Delta x]\frac{1}{2} f(\overline{x}_i) = \rho \cdot \frac{1}{2}[f(\overline{x}_i)]^2 \Delta x$$

Again we add these moments and take the limit to obtain the moment of $\mathcal{R}$ about the x-axis:

$$\boxed{M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \cdot \frac{1}{2}[f(\overline{x}_i)]^2 \Delta x = \rho \int_a^b \frac{1}{2}[f(x)]^2 dx}$$

Just as for systems of particles, the center of mass of the plate is defined so that $m\overline{x} = M_y$ and $m\overline{y} = M_x$. But the mass of the plate is the product of its density and its area:

$$m = \rho A = \rho \int_a^b f(x)dx$$

and so

$$\overline{x} = \frac{M_y}{m} = \frac{\rho \displaystyle\int_a^b x f(x)dx}{\rho \displaystyle\int_a^b f(x)dx} = \frac{\displaystyle\int_a^b x f(x)dx}{\displaystyle\int_a^b f(x)dx}$$

$$\overline{y} = \frac{M_x}{m} = \frac{\rho \displaystyle\int_a^b \frac{1}{2}[f(x)]^2 dx}{\rho \displaystyle\int_a^b f(x)dx} = \frac{\displaystyle\int_a^b \frac{1}{2}[f(x)]^2 dx}{\displaystyle\int_a^b f(x)dx}$$

Notice the cancellation of the ρ's. The location of the center of mass is independent of the density.

In summary, the center of mass of the plate (or the centroid of $\mathcal{R}$) is located at the point $(\overline{x}, \overline{y})$, where

$$\boxed{\overline{x} = \frac{1}{A}\int_a^b x f(x)dx \qquad \overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2 dx} \tag{12}$$

EXAMPLE: Find the center of mass of a semicircular plate of radius r.

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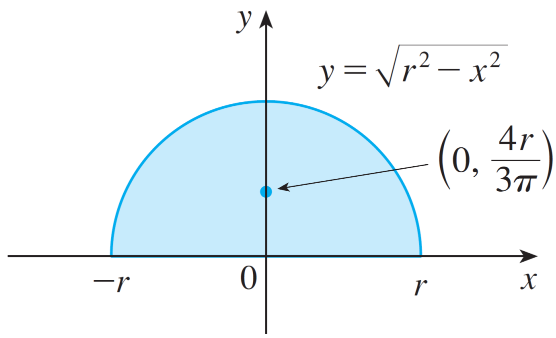

Solution: In order to use (12) we place the semicircle as in the figure above so that $f(x) = \sqrt{r^2 - x^2}$ and $a = -r,\ b = r$. Here there is no need to use the formula to calculate $\overline{x}$ because, by the symmetry principle, the center of mass must lie on the y-axis, so $\overline{x} = 0$. The area of the semicircle is $A = \frac{1}{2}\pi r^2$, so

$$\begin{aligned}
\overline{y} &= \frac{1}{A}\int_{-r}^{r}\frac{1}{2}[f(x)]^2dx = \frac{1}{\frac{1}{2}\pi r^2}\cdot\frac{1}{2}\int_{-r}^{r}(\sqrt{r^2 - x^2})^2dx \\
&= \frac{2}{\pi r^2}\int_0^r (r^2 - x^2)dx = \frac{2}{\pi r^2}\left[r^2x - \frac{x^3}{3}\right]_0^r \\
&= \frac{2}{\pi r^2}\cdot\frac{2r^3}{3} = \frac{4r}{3\pi}
\end{aligned}$$

The center of mass is located at the point $\left(0, \frac{4r}{3\pi}\right)$.

EXAMPLE: Find the centroid of the region bounded by the curves $y = \cos x,\ y = 0,\ x = 0$, and $x = \pi/2$.

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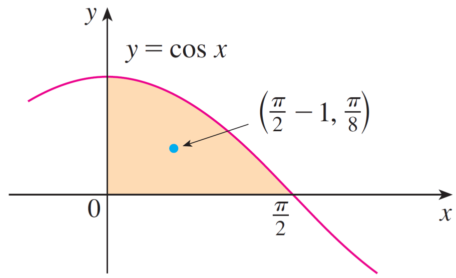


Solution: The area of the region is

$$A = \int_0^{\pi/2} \cos x dx = \sin x]_0^{\pi/2} = 1$$

so Formulas 12 give

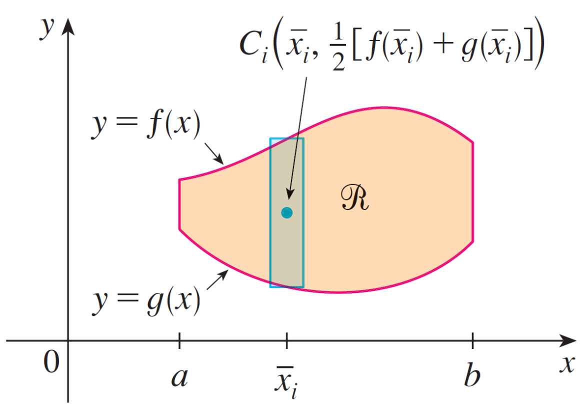
$$\overline{x} = \frac{1}{A}\int_0^{\pi/2} xf(x)dx = \int_0^{\pi/2} x\cos x dx = x\sin x]_0^{\pi/2} - \int_0^{\pi/2} \sin x dx = \frac{\pi}{2} - 1$$

and

$$\overline{y} = \frac{1}{A}\int_0^{\pi/2} \frac{1}{2}[f(x)]^2 dx = \frac{1}{2}\int_0^{\pi/2} \cos^2 x dx = \frac{1}{4}\int_0^{\pi/2} (1+\cos 2x)dx = \frac{1}{4}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/2} = \frac{\pi}{8}$$

The centroid is $\left(\frac{1}{2}\pi - 1,\ \frac{1}{8}\pi\right)$.

If the region $\mathcal{R}$ lies between two curves $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$, then the same sort of



argument that led to Formulas 12 can be used to show that the centroid of $\mathcal{R}$ is $(\overline{x}, \overline{y})$, where

$$\boxed{\overline{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]dx \qquad \overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left\{[f(x)]^2 - [g(x)]^2\right\} dx} \tag{13}$$

EXAMPLE: Find the centroid of the region bounded by the line $y = x$ and the parabola $y = x^2$.

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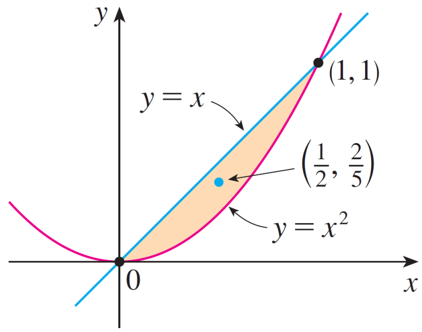

Solution: We take $f(x) = x,\ g(x) = x^2,\ a = 0$, and $b = 1$ in Formulas 13. First we note that the area of the region is

$$A = \int_0^1 (x - x^2)dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$$

Therefore

$$\overline{x} = \frac{1}{A}\int_0^1 x[f(x) - g(x)]dx = \frac{1}{1/6}\int_0^1 x(x - x^2)dx = 6\int_0^1 (x^2 - x^3)dx = 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2}$$

and

$$\overline{y} = \frac{1}{A}\int_0^1 \frac{1}{2}\left\{[f(x)]^2 - [g(x)]^2\right\}dx = \frac{1}{1/6}\int_0^1 \frac{1}{2}(x^2 - x^4)dx = 3\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{2}{5}$$

The centroid is $\left(\frac{1}{2},\ \frac{2}{5}\right)$.

We end this section by showing a surprising connection between centroids and volumes of revolution.

THEOREM OF PAPPUS: Let $\mathcal{R}$ be a plane region that lies entirely on one side of a line l in the plane. If $\mathcal{R}$ is rotated about l, then the volume of the resulting solid is the product of the area A of $\mathcal{R}$ and the distance d traveled by the centroid of $\mathcal{R}$.

Proof: We give the proof for the special case in which the region lies between $y = f(x)$ and $y = g(x)$ and the line l is the y-axis. Using the method of cylindrical shells we have

$$V = \int_a^b 2\pi x[f(x) - g(x)]dx = 2\pi\int_a^b x[f(x) - g(x)]dx \overset{(13)}{=} 2\pi(\overline{x}A) = (2\pi\overline{x})A = Ad$$

where $d = 2\pi\overline{x}$ is the distance traveled by the centroid during one rotation about the y-axis. ■

EXAMPLE: A torus is formed by rotating a circle of radius r about a line in the plane of the circle that is a distance $R\,(> r)$ from the center of the circle. Find the volume of the torus.

Solution: The circle has area $A = \pi r^2$. By the symmetry principle, its centroid is its center and so the distance traveled by the centroid during a rotation is $d = 2\pi R$. Therefore, by the Theorem of Pappus, the volume of the torus is

$$V = Ad = \left(\pi r^2\right)(2\pi R) = 2\pi^2 r^2 R$$