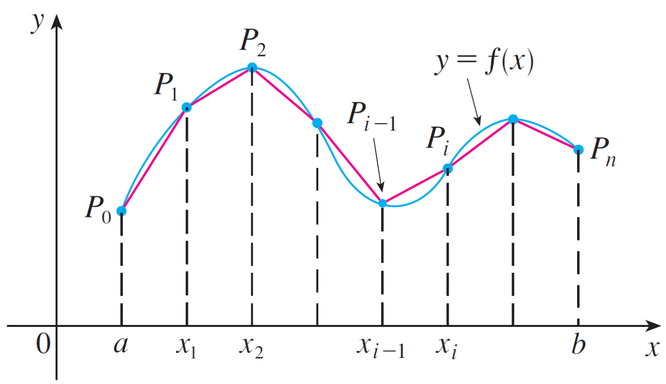
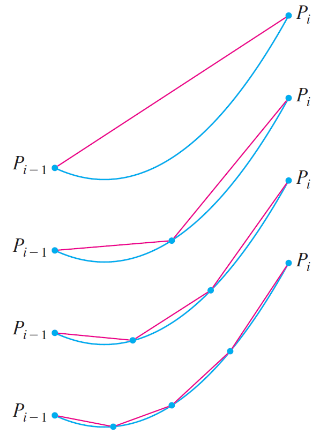
Arc Length

Suppose that a curve C is defined by the equation $y = f(x)$, where f is continuous and $a \leq x \leq b$. We obtain a polygonal approximation to C by dividing the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx. If $y_i = f(x_i)$, then the point $P_i(x_i, y_i)$ lies on C and the polygon with vertices $P_0, P_1, \ldots, P_n$ is an approximation to C.





The length L of C is approximately the length of this polygon and the approximation gets better as we let n increase. Therefore, we define the **length** L of the curve C with equation $y = f(x)$, $a \leq x \leq b$, as the limit of the lengths of the inscribed polygons with vertices $P_0, P_1, \ldots$ (if the limit exists):

$$\boxed{L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|} \tag{1}$$

Notice that the procedure for defining arc length is very similar to the procedure we used for defining area and volume: We divided the curve into a large number of small parts. We then found the approximate lengths of the small parts and added them. Finally, we took the limit as $n \to \infty$.

The definition of arc length given by Equation 1 is not very convenient for computational purposes, but we can derive an integral formula for L in the case where f has a continuous derivative. [Such a function f is called **smooth** because a small change in x produces a small change in $f'(x)$.]

If we let $\Delta y_i = y_i - y_{i-1}$, then

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

By applying the Mean Value Theorem to f on the interval $[x_{i-1},\ x_i]$, we find that there is a number x_i^* between x_{i-1} and x_i such that

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$

that is,

$$\Delta y_i = f'(x_i^*)\Delta x$$

Thus we have

$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\Delta y_i)^2} = \sqrt{(\Delta x)^2 + [f'(x_i^*)\Delta x]^2} = \sqrt{(\Delta x)^2 + [f'(x_i^*)]^2(\Delta x)^2}$$

$$= \sqrt{(1 + [f'(x_i^*)]^2)(\Delta x)^2} = \sqrt{1 + [f'(x_i^*)]^2}\sqrt{(\Delta x)^2} = \sqrt{1 + [f'(x_i^*)]^2}\Delta x$$

Therefore, by the Definition above,

$$L = \lim_{n\to\infty}\sum_{i=1}^{n}|P_{i-1}P_i| = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1+[f'(x_i^*)]^2}\Delta x$$

We recognize this expression as being equal to

$$\int_a^b \sqrt{1+[f'(x)]^2}dx$$

by the definition of a definite integral. This integral exists because the function $g(x) = \sqrt{1+[f'(x)]^2}$ is continuous. Thus we have proved the following theorem:

THEOREM: If f' is continuous on $[a,\ b]$, then the length of the curve $y = f(x), a \le x \le b$, is

$$\boxed{L = \int_a^b \sqrt{1+[f'(x)]^2}dx} \tag{2}$$

If we use Leibniz notation for derivatives, we can write the arc length formula as follows:

$$\boxed{L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx} \tag{3}$$

EXAMPLE: Find the length of the segment of the horizontal line $y = 1$ between the points $(a, 1)$ and $(b, 1)$.

Solution: Since $f(x) = 1$, we have $f'(x) = 0$, and (2) gives

$$L = \int_a^b \sqrt{1+[f'(x)]^2}dx = \int_a^b \sqrt{1+0}dx = \int_a^b dx = b - a$$

EXAMPLE: Find the length of the segment of the line $y = x$ between the points (a, a) and (b, b).

Solution: Since $f(x) = x$, we have $f'(x) = 1$, and (2) gives

$$L = \int_a^b \sqrt{1+[f'(x)]^2}dx = \int_a^b \sqrt{1+1}dx = \sqrt{2}\int_a^b dx = \sqrt{2}(b-a)$$

EXAMPLE: Find the length of the segment of the line $y = mx + n$ between the points $(a, f(a))$ and $(b, f(b))$.

Solution: Since $f(x) = mx + n$, we have $f'(x) = m$, and (2) gives

$$L = \int_a^b \sqrt{1+[f'(x)]^2}dx = \int_a^b \sqrt{1+m^2}dx = \sqrt{1+m^2}\int_a^b dx = \sqrt{1+m^2}(b-a)$$

EXAMPLE: Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$.

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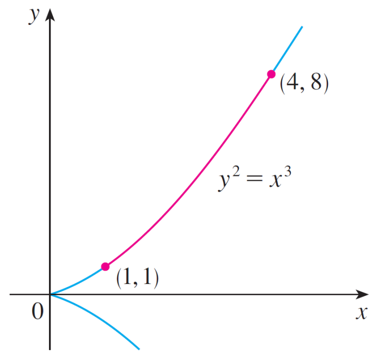

Solution: For the top half of the curve we have

$$y = x^{3/2} \qquad \frac{dy}{dx} = \frac{3}{2}x^{1/2}$$

and so the arc length formula gives

$$L = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^4 \sqrt{1 + \frac{9}{4}x dx}$$

If we substitute $u = 1 + \frac{9}{4}x$, then $du = \frac{9}{4}dx$. When $x = 1, u = \frac{13}{4}$; when $x = 4, u = 10$. Therefore

$$L = \frac{4}{9}\int_{13/4}^{10} \sqrt{u}du = \frac{4}{9} \cdot \frac{2}{3}u^{3/2}\Big]_{13/4}^{10} = \frac{8}{27}\left[10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right] = \frac{1}{27}(80\sqrt{10} - 13\sqrt{13})$$

If a curve has the equation $x = g(y)$, $c \le y \le d$, and $g'(y)$ is continuous, then by interchanging the roles of x and y in (2) and (3), we obtain the following formula for its length:

$$\boxed{L = \int_c^d \sqrt{1 + [g'(y)]^2}dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy} \tag{4}$$

EXAMPLE: Find the length of the arc of the parabola $y^2 = x$ from $(0, 0)$ to $(1, 1)$.

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Solution: Since $x = y^2$, we have $dx/dy = 2y$, and (4) gives

$$L = \int_0^1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_0^1 \sqrt{1+4y^2}\,dy$$

We make the trigonometric substitution $y = \frac{1}{2}\tan\theta$, which gives $dy = \frac{1}{2}\sec^2\theta\,d\theta$ and $\sqrt{1+4y^2} = \sqrt{1+\tan^2\theta} = \sec\theta$. When $y = 0$, $\tan\theta = 0$, so $\theta = 0$; when $y = 1$, $\tan\theta = 2$, so $\theta = \tan^{-1} 2 = \alpha$, say. Thus

$$\begin{aligned} L &= \int_0^\alpha \sec\theta \cdot \frac{1}{2}\sec^2\theta\,d\theta = \frac{1}{2}\int_0^\alpha \sec^3\theta\,d\theta \\ &= \frac{1}{2}\cdot\frac{1}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_0^\alpha \\ &= \frac{1}{4}(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|) \end{aligned}$$

(We could have used Formula 21 in the Table of Integrals.) Since $\tan\alpha = 2$, we have $\sec^2\alpha = 1+\tan^2\alpha = 5$, so $\sec\alpha = \sqrt{5}$ and

$$L = \frac{\sqrt{5}}{2} + \frac{\ln(\sqrt{5}+2)}{4}$$

Because of the presence of the square root sign in Formulas 2 and 4, the calculation of an arc length often leads to an integral that is very difficult or even impossible to evaluate explicitly. Thus we sometimes have to be content with finding an approximation to the length of a curve, as in the following example.

EXAMPLE:

(a) Set up an integral for the length of the arc of the hyperbola $xy = 1$ from the point $(1, 1)$ to the point $\left(2, \frac{1}{2}\right)$.

(b) Use Simpson's Rule with $n = 10$ to estimate the arc length.

Solution:

(a) We have

$$y = \frac{1}{x} \qquad \frac{dy}{dx} = -\frac{1}{x^2}$$

and so the arc length is

$$L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_1^2 \sqrt{1 + \frac{1}{x^4}}\,dx$$

(b) Using Simpson's Rule (see Section 7.7) with $a = 1, b = 2, n = 10, \Delta x = 0.1$, and $f(x) = \sqrt{1 + 1/x^4}$, we have

$$L = \int_1^2 \sqrt{1+\frac{1}{x^4}}\,dx \approx \frac{\Delta x}{3}[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + \ldots + 2f(1.8) + 4f(1.9) + f(2)] \approx 1.1321$$

The Arc Length Function

We will find it useful to have a function that measures the arc length of a curve from a particular starting point to any other point on the curve. Thus if a smooth curve C has the equation $y = f(x), a \leq x \leq b$, let $s(x)$ be the distance along C from the initial point $P_0(a, f(a))$ to the point $Q(x, f(x))$. Then s is a function, called the **arc length function**, and, by (2),

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2} dt \tag{5}$$

We can use Part 1 of the Fundamental Theorem of Calculus to differentiate (5) (since the integrand is continuous):

$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{6}$$

The last Equation shows that the rate of change of s with respect to x is always at least 1 and is equal to 1 when $f'(x)$, the slope of the curve, is 0. The differential of arc length is

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \tag{7}$$

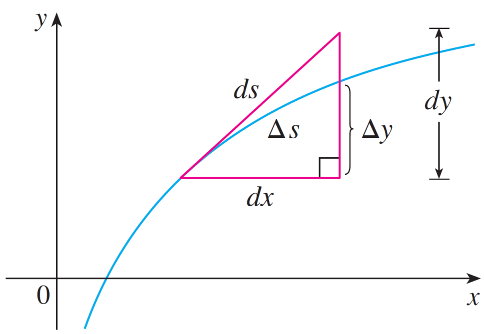
and this equation is sometimes written in the symmetric form

$$(ds)^2 = (dx)^2 + (dy)^2 \tag{8}$$

The geometric interpretation of Equation 8 is shown in the Figure on the right. If we write $L = \int ds$, then from (8) either we can solve to get (7), which gives (3), or we can solve to get

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

which gives (4).



EXAMPLE: Find the arc length function for the curve $y = x^2 - \frac{1}{8} \ln x$ taking $P_0(1, 1)$ as the starting point.

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Solution: If $f(x) = x^2 - \frac{1}{8}\ln x$, then

$$f'(x) = 2x - \frac{1}{8x}$$

$$1 + [f'(x)]^2 = 1 + \left(2x - \frac{1}{8x}\right)^2 = 1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2} = 4x^2 + \frac{1}{2} + \frac{1}{64x^2} = \left(2x + \frac{1}{8x}\right)^2$$

$$\sqrt{1 + [f'(x)]^2} = 2x + \frac{1}{8x}$$

Thus the arc length function is given by

$$s(x) = \int_1^x \sqrt{1 + [f'(t)]^2}dt = \int_1^x \left(2t + \frac{1}{8t}\right) dt = t^2 + \frac{1}{8}\ln t\Big]_1^x = x^2 + \frac{1}{8}\ln x - 1$$

For instance, the arc length along the curve from $(1,1)$ to $(3, f(3))$ is

$$s(3) = 3^2 + \frac{1}{8}\ln 3 - 1 = 8 + \frac{\ln 3}{8} \approx 8.1373$$