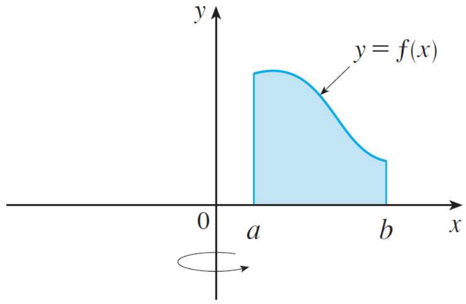
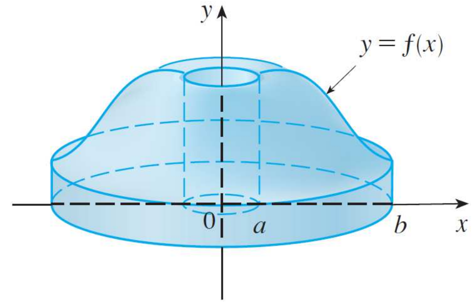
Volumes by Cylindrical Shells

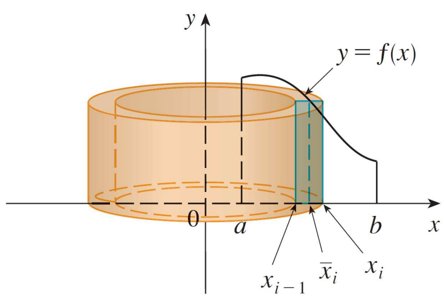
Let S be the solid obtained by rotating about the y-axis the region bounded by $y = f(x)$ [where f is continuous and $f(x) \geq 0$], $y = 0$, $x = a$, and $x = b$, where $b > a \geq 0$.

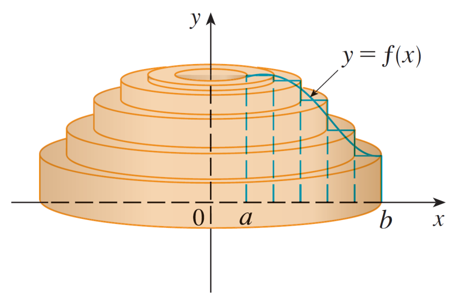




The volume of the solid in the figure above, obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b, is

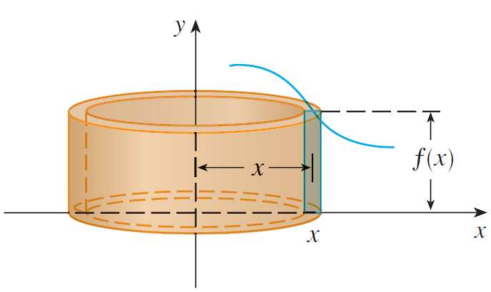
$$V = \int_a^b 2\pi x f(x) dx \quad \text{where} \quad 0 \leq a < b$$

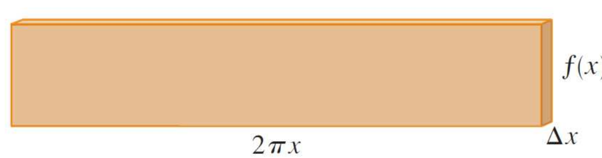




The best way to remember the above formula is to think of a typical shell, cut and flattened as in the figure below, with radius x, circumference $2\pi x$, height $f(x)$, and thickness Δx or dx:

$$\int_a^b \underbrace{(2\pi x)}_{\text{circumference}} \underbrace{[f(x)]}_{\text{height}} dx$$





EXAMPLE: Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and the x-axis.

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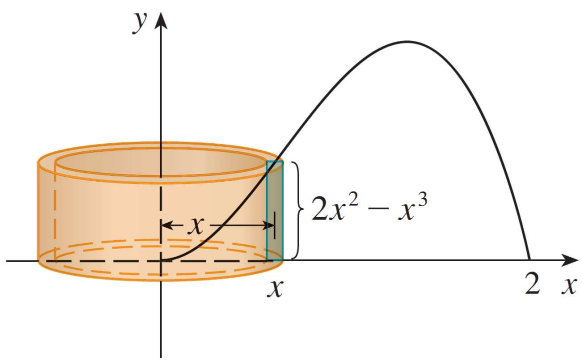

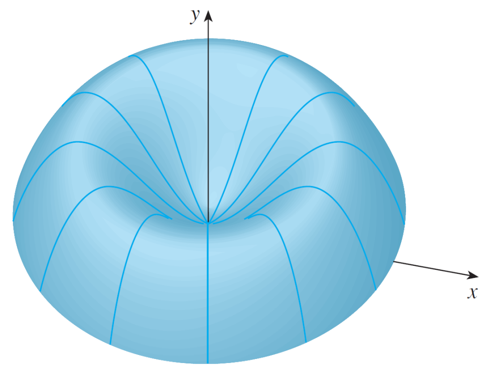

Solution: We have

$$2\pi(\text{radius})(\text{height}) = 2\pi \cdot x \cdot (2x^2 - x^3) = 2\pi(2x^3 - x^4)$$

Note that $2x^2 - x^3 = 0$ if $x = 0, 2$. Therefore

$$V = \int_0^2 2\pi(2x^3 - x^4)dx = 2\pi \int_0^2 (2x^3 - x^4)dx = 2\pi \left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2 = \frac{16\pi}{5}$$

EXAMPLE: Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = (x-1)(x-3)^2$ and the x-axis.

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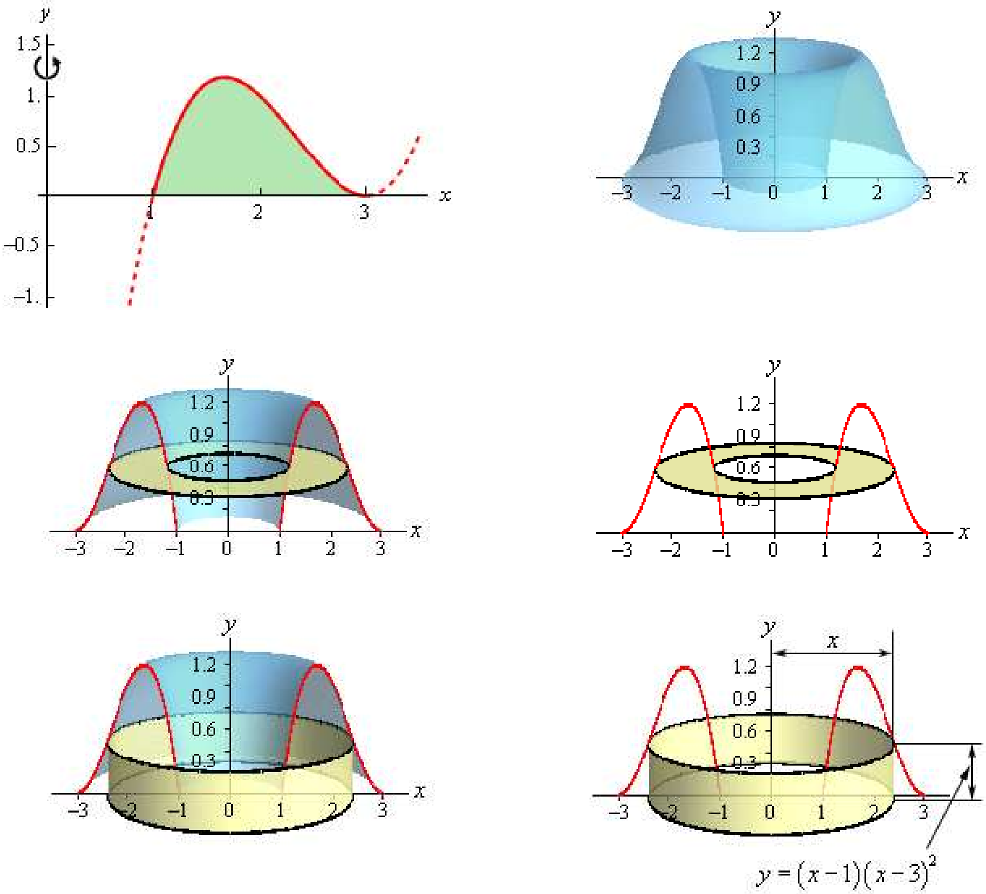

Solution: We have

$$2\pi(\text{radius})(\text{height}) = 2\pi \cdot x \cdot (x-1)(x-3)^2 = 2\pi(x^4 - 7x^3 + 15x^2 - 9x)$$

Note that $(x-1)(x-3)^2 = 0$ if $x = 1, 3$. Therefore

$$V = \int_1^3 2\pi(x^4 - 7x^3 + 15x^2 - 9x)dx$$

$$= 2\pi \int_1^3 (x^4 - 7x^3 + 15x^2 - 9x)dx = 2\pi \left[\frac{x^5}{5} - \frac{7x^4}{4} + 5x^3 - \frac{9x^2}{2}\right]_1^3 = \frac{24\pi}{5}$$

EXAMPLE: Find the volume of the solid obtained by rotating about the x-axis the region bounded by $y = \sqrt[3]{x},\ x = 8$ and the x-axis.

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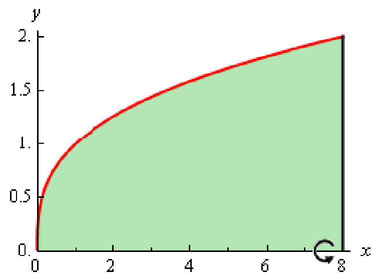

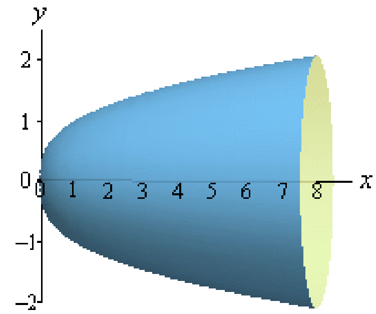

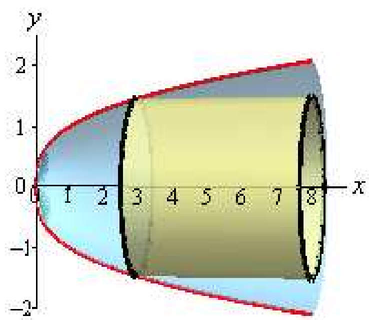

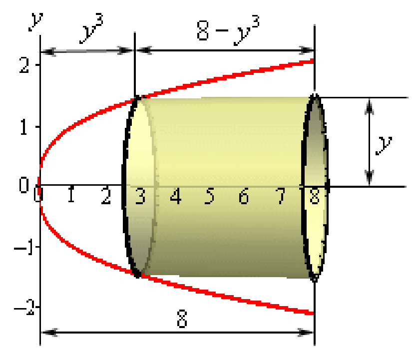

Solution 1 (Discs): We have

$$A(x) = \pi(x^{1/3})^2 = \pi x^{2/3}$$

therefore

$$V = \int_0^8 A(x)dx = \int_0^8 \pi x^{2/3}dx = \pi \left.\frac{x^{2/3+1}}{2/3+1}\right]_0^8 = \pi \left.\frac{3}{5}x^{5/3}\right]_0^8 = \pi\frac{3}{5}8^{5/3} = \frac{96\pi}{5}$$

Solution 2 (Shells): We have

$$y = \sqrt[3]{x} \implies x = y^3$$

hence

$$2\pi(\text{radius})(\text{width}) = 2\pi \cdot y \cdot (8 - y^3) = 2\pi(8y - y^4)$$

Note that $8y - y^4 = 0$ if $y = 0, 2$. Therefore

$$V = \int_0^2 2\pi(8y - y^4)dy = 2\pi \int_0^2 (8y - y^4)dy = 2\pi \left[4y^2 - \frac{y^5}{5}\right]_0^2 = \frac{96\pi}{5}$$

EXAMPLE: Find the volume of the solid obtained by rotating about the line $x = 6$ the region bounded by $y = 2\sqrt{x-1}$ and $y = x - 1$.

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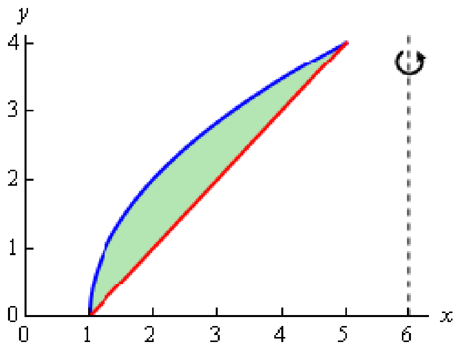

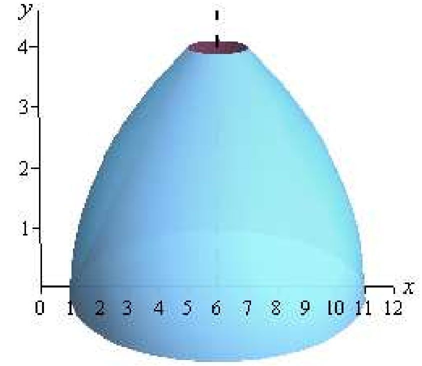

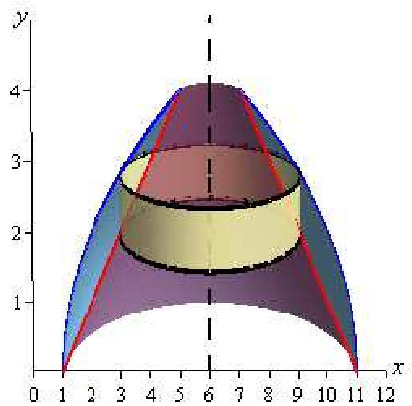

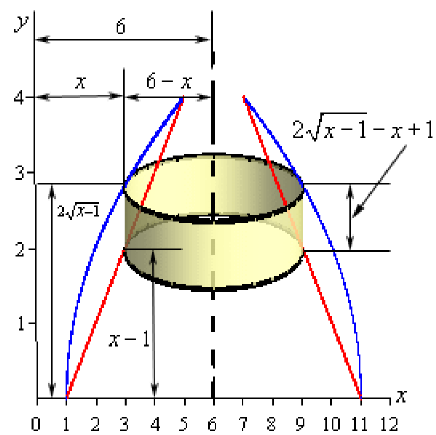

Solution 1 (Shells): We have

$$\begin{aligned} 2\pi(\text{radius})(\text{height}) &= 2\pi \cdot (6-x)\cdot(2\sqrt{x-1} - x + 1) \\ &= 2\pi(x^2 - 7x + 6 + 12\sqrt{x-1} - 2x\sqrt{x-1}) \end{aligned}$$

Note that $2\sqrt{x-1} = x - 1$ if $x = 1, 5$. Therefore

$$\begin{aligned} V &= \int_1^5 2\pi(x^2 - 7x + 6 + 12\sqrt{x-1} - 2x\sqrt{x-1})dx \\ &= 2\pi\int_1^5 (x^2 - 7x + 6 + 12\sqrt{x-1} - 2x\sqrt{x-1})dx \\ &= 2\pi\left[\frac{x^3}{3} - \frac{7x^2}{2} + 6x + 8(x-1)^{3/2} - \frac{4}{3}(x-1)^{3/2} - \frac{4}{5}(x-1)^{5/2}\right]_1^5 \\ &= \frac{272\pi}{15} \end{aligned}$$

Solution 2 (Discs): We have

$$y = 2\sqrt{x-1} \implies x = \frac{y^2}{4} + 1$$

$$y = x - 1 \implies x = y + 1$$

hence

$$\begin{aligned} A(y) = \pi R_{big}^2 - \pi R_{small}^2 = \pi(R_{big}^2 - R_{small}^2) &= \pi\left(\left[6 - \left(\frac{y^2}{4} + 1\right)\right]^2 - [6 - (y+1)]^2\right) \\ &= \pi\left(\left(5 - \frac{y^2}{4}\right)^2 - (5-y)^2\right) \\ &= \pi\left(\frac{1}{16}y^4 - \frac{7}{2}y^2 + 10y\right) \end{aligned}$$

Note that $\frac{y^2}{4} + 1 = y + 1$ if $y = 0, 4$. Therefore

$$V = \int_0^4 A(y)dy = \int_0^4 \pi\left(\frac{1}{16}y^4 - \frac{7}{2}y^2 + 10y\right)dy = \pi\left[\frac{y^5}{80} - \frac{7y^3}{6} + 5y^2\right]_0^4 = \frac{272\pi}{15}$$

EXAMPLE: Find the volume of the solid obtained by rotating about the line $y = -1$ the region bounded by $x = (y-2)^2$ and $y = x$.

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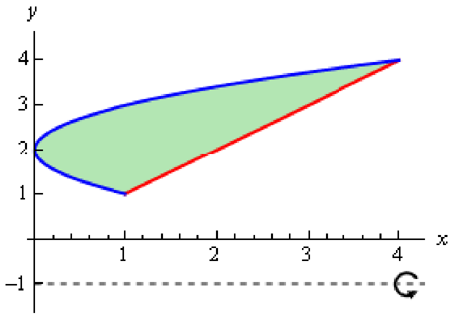

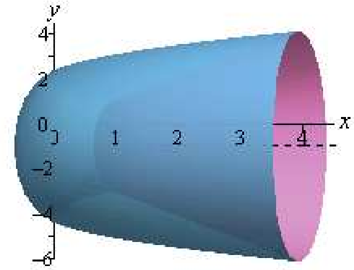

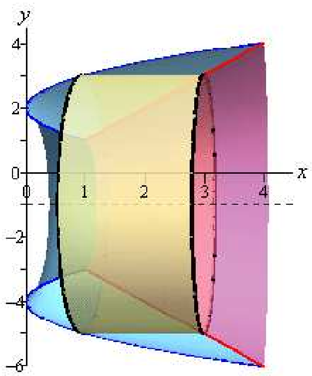

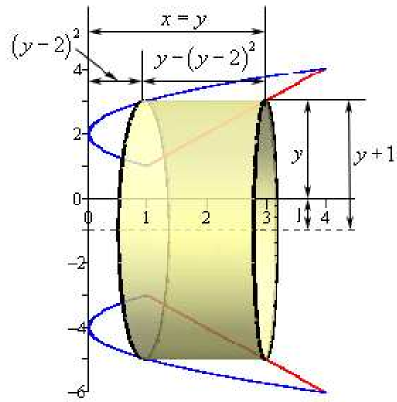

Solution: We have

$$2\pi(\text{radius})(\text{width}) = 2\pi \cdot (y+1) \cdot (y - (y-2)^2) = 2\pi(-y^3 + 4y^2 + y - 4)$$

Note that $(y-2)^2 = y$ if $y = 1, 4$. Therefore

$$\begin{aligned} V &= \int_1^4 2\pi(-y^3 + 4y^2 + y - 4)dy \\ &= 2\pi \int_1^4 (-y^3 + 4y^2 + y - 4)dy \\ &= 2\pi \left[-\frac{y^4}{4} + \frac{4y^3}{3} + \frac{y^2}{2} - 4y \right]_1^4 \\ &= \frac{63\pi}{2} \end{aligned}$$