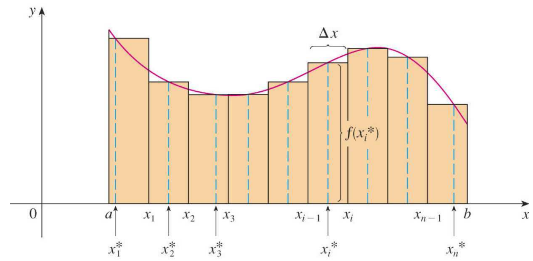
Volumes

DEFINITION OF A DEFINITE INTEGRAL: If f is a function defined on $[a, b]$, the **definite integral** of f from a to b is a number

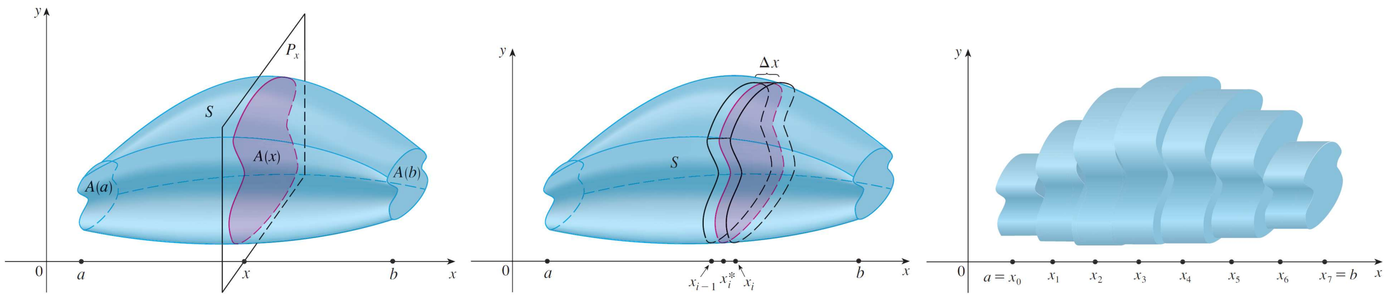
$$\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

provided that this limit exists.

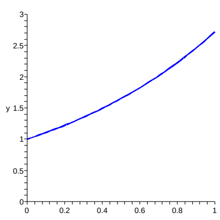


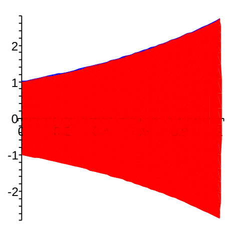
DEFINITION OF VOLUME: Let S be a solid that lies between $x = a$ and $x = b$. If the cross-section area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where $A(x)$ is an integrable function, then the **volume** of S is

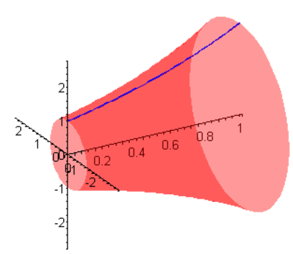
$$V = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} A(x_i^*)\Delta x_i = \int_a^b A(x)dx$$

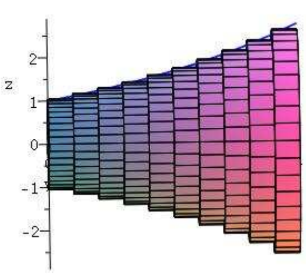


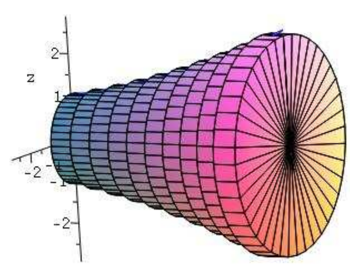
EXAMPLE: Find the volume of the solid obtaining by rotating about the x-axis the region under the curve $y = e^x$ from 0 to 1.







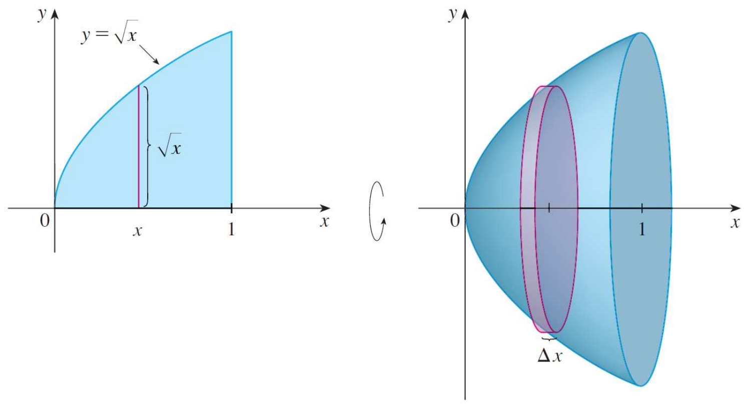




Solution: We have

$$A(x) = \pi R^2 = \pi(e^x)^2 = \pi e^{2x} \quad \Longrightarrow \quad V = \int_0^1 A(x)dx = \int_0^1 \pi e^{2x}dx = \pi\int_0^1 e^{2x}dx = \pi\frac{1}{2}e^{2x}\Big]_0^1 = \frac{\pi}{2}(e^2 - 1)$$

EXAMPLE: Find the volume of the solid obtaining by rotating about the x-axis the region under the curve $y=\sqrt{x}$ from 0 to 1.

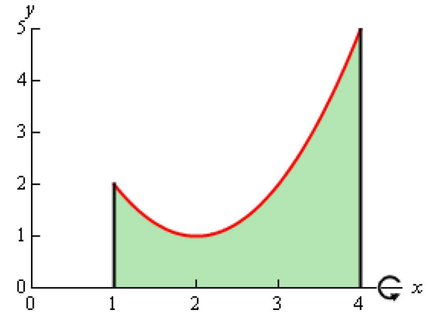


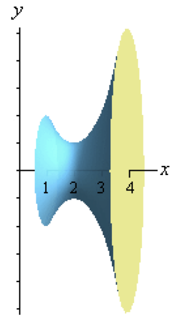
Solution: We have

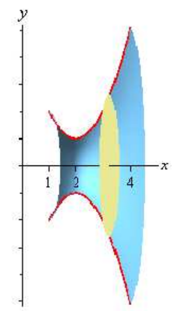
$$A(x)=\pi R^2=\pi(\sqrt{x})^2=\pi x \quad \Longrightarrow \quad V=\int_0^1 A(x)dx=\int_0^1 \pi x dx=\pi\int_0^1 x dx=\pi\frac{x^2}{2}\bigg]_0^1=\frac{\pi}{2}$$

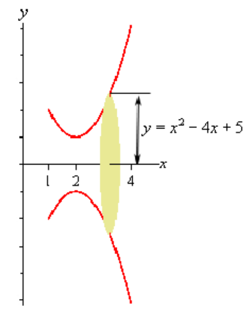
EXAMPLE: Find the volume of the solid obtaining by rotating about the x-axis the region bounded by $y=x^2-4x+5,\ x=1$ and $x=4$.

EXAMPLE: Find the volume of the solid obtaining by rotating about the x-axis the region bounded by $y = x^2 - 4x + 5$, $x = 1$ and $x = 4$.









Solution: We have

$$A(x) = \pi(x^2 - 4x + 5)^2 = \pi(x^4 - 8x^3 + 26x^2 - 40x + 25)$$

therefore

$$V = \int_1^4 A(x)dx = \int_1^4 \pi(x^4 - 8x^3 + 26x^2 - 40x + 25)dx$$

$$= \pi \int_1^4 (x^4 - 8x^3 + 26x^2 - 40x + 25)dx = \pi \left[\frac{x^5}{5} - 2x^4 + \frac{26x^3}{3} - 20x^2 + 25x\right]_1^4 = \frac{78\pi}{5}$$

EXAMPLE: Show that the volume of a cone of height h and radius r is $V = \frac{1}{3}\pi r^2 h$.

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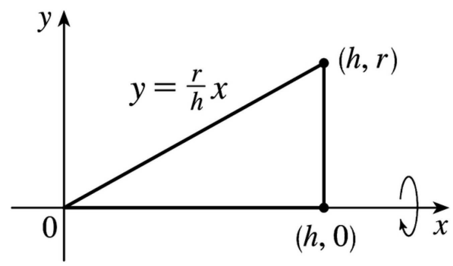
Solution: We have

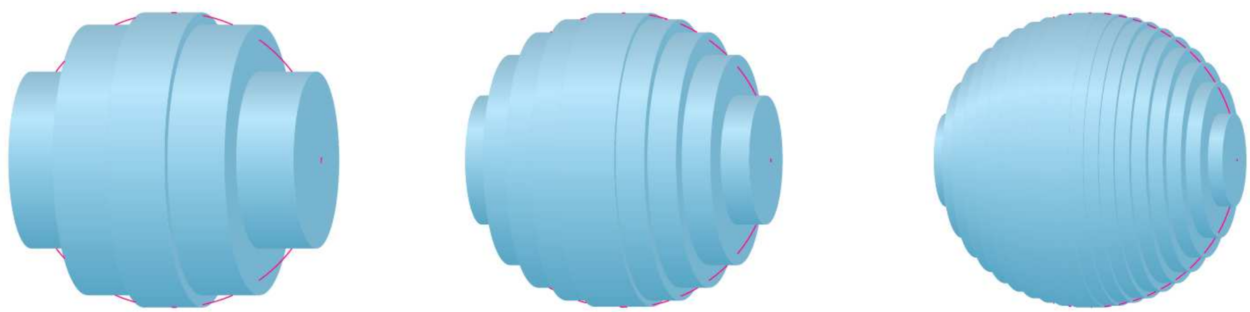
$$A(x) = \pi R^2 = \pi \left(\frac{rx}{h}\right)^2 = \pi \frac{r^2x^2}{h^2}$$



therefore

$$\begin{aligned} V &= \int_0^h A(x)dx = \int_0^h \pi \frac{r^2x^2}{h^2}dx = \pi\frac{r^2}{h^2}\int_0^h x^2dx \\ &= \pi\frac{r^2}{h^2}\cdot\frac{x^3}{3}\Big]_0^h = \pi\frac{r^2}{h^2}\cdot\frac{h^3}{3} = \frac{1}{3}\pi r^2 h \end{aligned}$$

EXAMPLE: Show that the volume of a sphere of radius r is $V = \dfrac{4}{3}\pi r^3$.



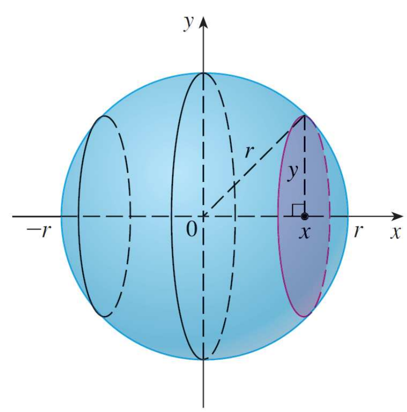
(a) Using 5 disks, $V \approx 4.2726$ (b) Using 10 disks, $V \approx 4.2097$ (c) Using 20 disks, $V \approx 4.1940$

Solution: We have

$$A(x) = \pi R^2 = \pi\left(\sqrt{r^2 - x^2}\right)^2 = \pi(r^2 - x^2)$$

therefore

$$\begin{aligned} V &= \int_{-r}^{r} A(x)dx = \int_{-r}^{r}\pi(r^2 - x^2)dx = \pi\int_{-r}^{r}(r^2 - x^2)dx \\ &= \pi\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r} = \pi\left(r^2\cdot r - \frac{r^3}{3}\right) - \pi\left(r^2\cdot(-r) - \frac{(-r)^3}{3}\right) \\ &= \frac{4}{3}\pi r^3 \end{aligned}$$



EXAMPLE: Find the volume of the solid obtaining by rotating about the y-axis the region bounded by $y = x^3,\ y = 8$, and $x = 0$.

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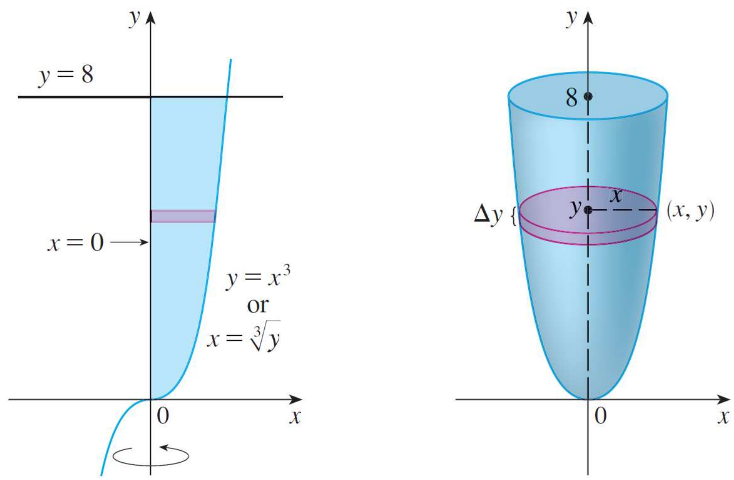

Solution: We have

$$A(y) = \pi R^2 = \pi(\sqrt[3]{y})^2 = \pi y^{2/3}$$

therefore

$$V = \int_0^8 A(y)dy = \int_0^8 \pi y^{2/3}dy = \pi \int_0^8 y^{2/3}dy = \pi \frac{y^{2/3+1}}{2/3+1}\Bigg]_0^8 = \pi \frac{3}{5}y^{5/3}\Bigg]_0^8 = \frac{96\pi}{5}$$

EXAMPLE: Find the volume of the solid obtaining by rotating about the x-axis the region bounded by $y = x$ and $y = x^2$.

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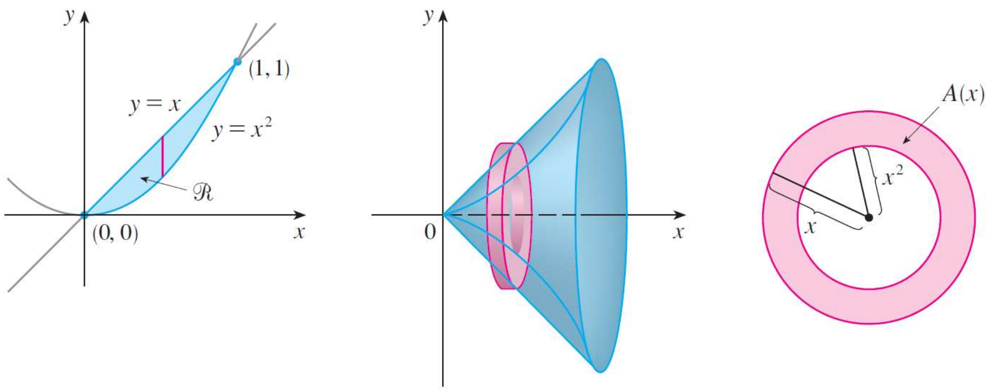


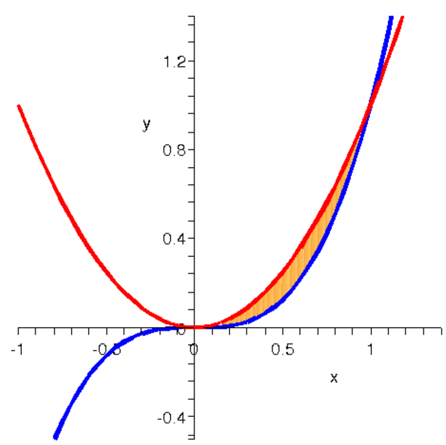
Solution: We have

$$A(x) = \pi R_{big}^2 - \pi R_{small}^2 = \pi(R_{big}^2 - R_{small}^2) = \pi(x^2 - (x^2)^2) = \pi(x^2 - x^4)$$

Note that $x = x^2$ if $x = 0, 1$. Therefore

$$V = \int_0^1 A(x)dx = \int_0^1 \pi(x^2 - x^4)dx = \pi \int_0^1 (x^2 - x^4)dx = \pi \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$$

EXAMPLE: Find the volume of the solid obtaining by rotating about the x-axis the region bounded by $y = x^3$ and $y = x^2$.



Solution: We have

$$A(x) = \pi R_{big}^2 - \pi R_{small}^2 = \pi(R_{big}^2 - R_{small}^2) = \pi((x^2)^2 - (x^3)^2) = \pi(x^4 - x^6)$$

Note that $x^3 = x^2$ if $x = 0, 1$. Therefore

$$V = \int_0^1 A(x)dx = \int_0^1 \pi(x^4 - x^6)dx = \pi \int_0^1 (x^4 - x^6)dx = \pi \left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^1 = \pi\left(\frac{1}{5} - \frac{1}{7}\right) = \frac{2\pi}{35}$$

EXAMPLE: Find the volume of the solid obtaining by rotating about the y-axis the region bounded by $y = \sqrt[3]{x}$ and $y = \dfrac{x}{4}$, where $x, y \geq 0$.

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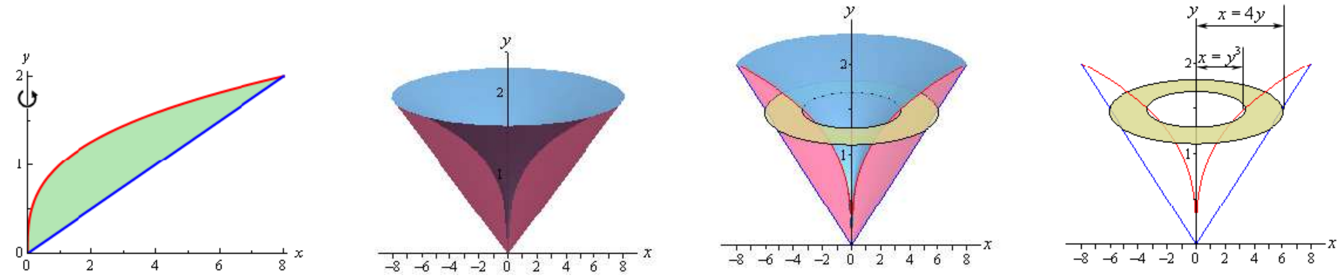

Solution: We have

$$y=\sqrt[3]{x} \implies x=y^3$$

$$y=\frac{x}{4} \implies x=4y$$

therefore

$$A(y)=\pi R_{big}^2-\pi R_{small}^2=\pi(R_{big}^2-R_{small}^2)=\pi((4y)^2-(y^3)^2)=\pi(16y^2-y^6)$$

Note that $y^3=4y$ if $y=0,\pm 2$. We exclude the negative root, since $y\ge 0$. We get

$$V=\int_0^2 A(y)dy=\int_0^2 \pi(16y^2-y^6)dy=\pi\int_0^2(16y^2-y^6)dy=\pi\left[\frac{16y^3}{3}-\frac{y^7}{7}\right]_0^2=\frac{512\pi}{21}$$

EXAMPLE: Find the volume of the solid obtaining by rotating about the line $y=2$ the region bounded by $y=x$ and $y=x^2$.

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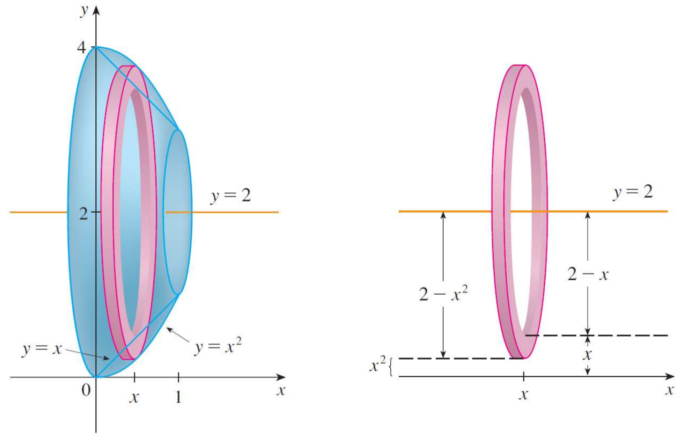


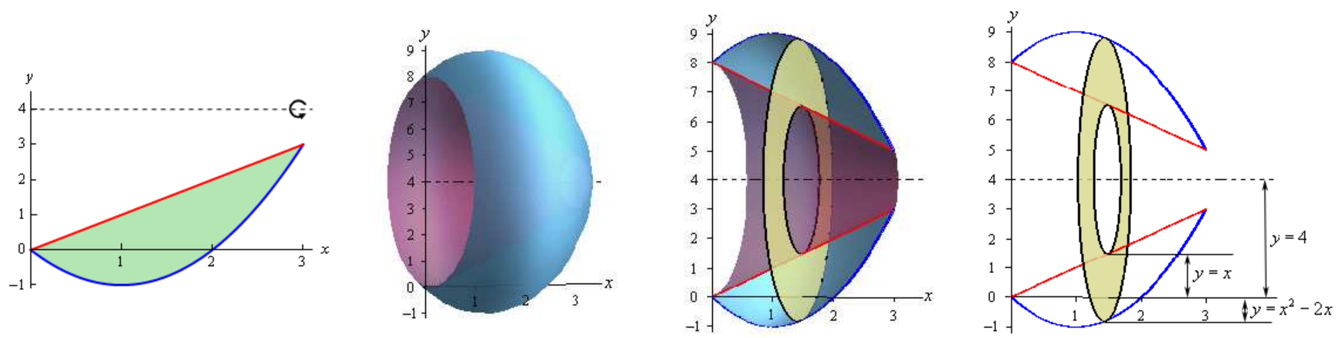
Solution: We have

$$A(x) = \pi(2 - x^2)^2 - \pi(2 - x)^2$$

Note that $x = x^2$ if $x = 0, 1$. Therefore

$$V = \int_0^1 A(x)dx = \int_0^1 [\pi(2 - x^2)^2 - \pi(2 - x)^2]dx = \pi \int_0^1 (x^4 - 5x^2 + 4x)dx = \pi \left[\frac{x^5}{5} - 5\frac{x^3}{3} + 4\frac{x^2}{2}\right]_0^1 = \frac{8\pi}{15}$$

EXAMPLE: Find the volume of the solid obtaining by rotating about the line $y = 4$ the region bounded by $y = x^2 - 2x$ and $y = x$.



Solution: We have

$$A(x) = \pi R_{big}^2 - \pi R_{small}^2 = \pi(R_{big}^2 - R_{small}^2) = \pi((4 - (x^2 - 2x))^2 - (4 - x)^2) = \pi(x^4 - 4x^3 - 5x^2 + 24x)$$

Note that $x^2 - 2x = x$ if $x = 0, 3$. Therefore

$$V = \int_0^3 A(x)dx = \int_0^3 \pi(x^4 - 4x^3 - 5x^2 + 24x)dx$$

$$= \pi \int_0^3 (x^4 - 4x^3 - 5x^2 + 24x)dx = \pi \left[\frac{x^5}{5} - x^4 - \frac{5x^3}{3} + 12x\right]_0^3 = \frac{153\pi}{5}$$

EXAMPLE: Find the volume of the solid obtaining by rotating about the line $x = -1$ the region bounded by $y = 2\sqrt{x - 1}$ and $y = x - 1$.

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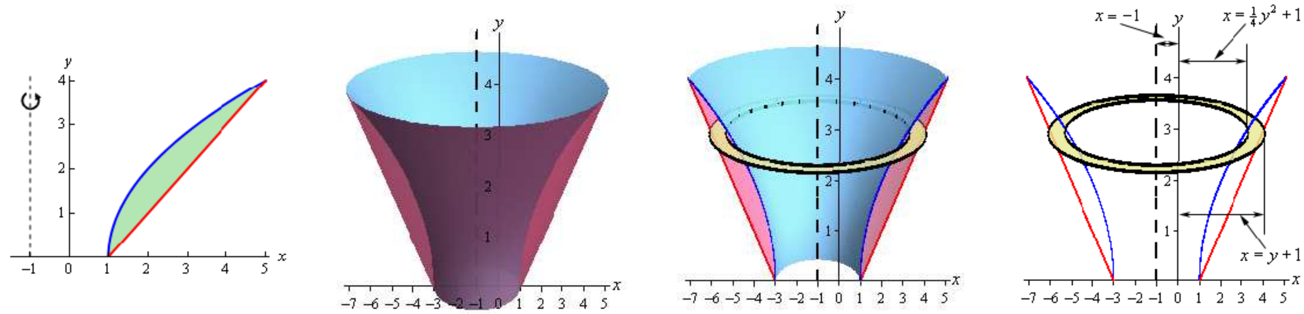


Solution: We have

$$y=2\sqrt{x-1} \implies x=\frac{y^2}{4}+1$$

$$y=x-1 \implies x=y+1$$

therefore

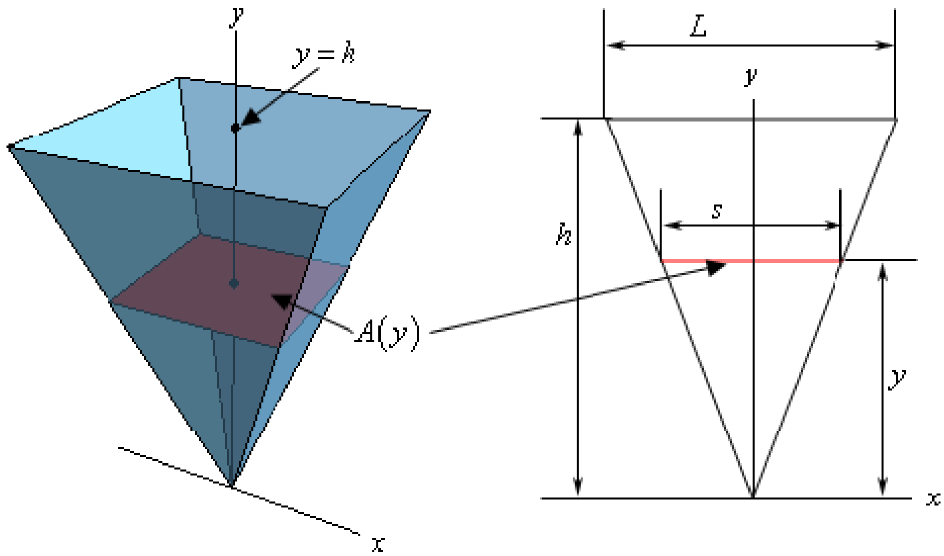
$$A(y)=\pi R_{big}^2-\pi R_{small}^2=\pi(R_{big}^2-R_{small}^2)=\pi\left((y+1+1)^2-\left(\frac{y^2}{4}+1+1\right)^2\right)=\pi\left(4y-\frac{y^4}{16}\right)$$

Note that $\frac{y^2}{4}+1=y+1$ if $y=0,4$. Hence

$$\begin{aligned} V &= \int_0^4 A(y)dy=\int_0^4 \pi\left(4y-\frac{y^4}{16}\right)dy \\ &= \pi\int_0^4\left(4y-\frac{y^4}{16}\right)dy=\pi\left[2y^2-\frac{y^5}{80}\right]_0^4=\frac{96\pi}{5} \end{aligned}$$

EXAMPLE: Find the volume of a pyramid whose base is a square with sides of length L and whose height is h.

EXAMPLE: Find the volume of a pyramid whose base is a square with sides of length L and whose height is h.



Solution: We have

$$\frac{s}{L}=\frac{y}{h} \quad \Longrightarrow \quad s=\frac{L}{h}y$$

therefore

$$A(y)=s^2=\frac{L^2}{h^2}y^2$$

hence

$$V=\int_0^h A(y)dy=\int_0^h \frac{L^2}{h^2}y^2dy=\frac{L^2}{h^2}\int_0^h y^2dy=\frac{L^2}{h^2}\cdot\frac{y^3}{3}\bigg]_0^h=\frac{1}{3}L^2h$$