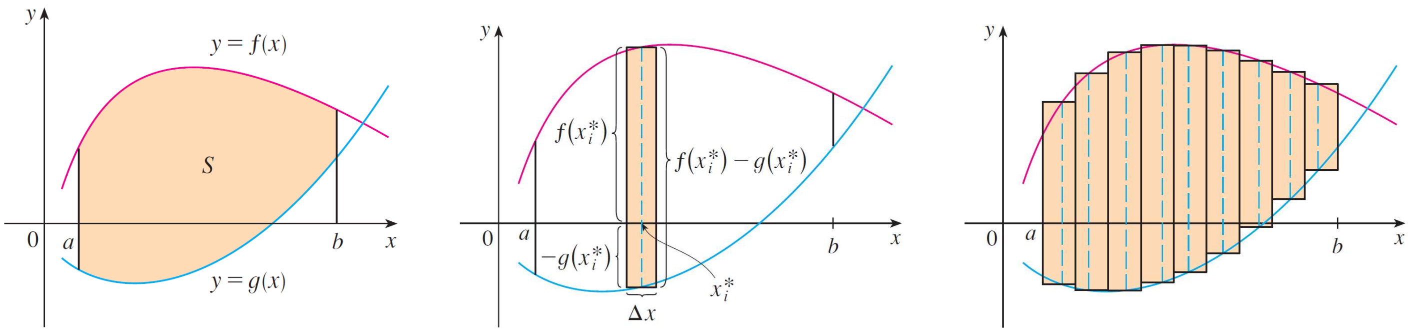
Areas Between Curves



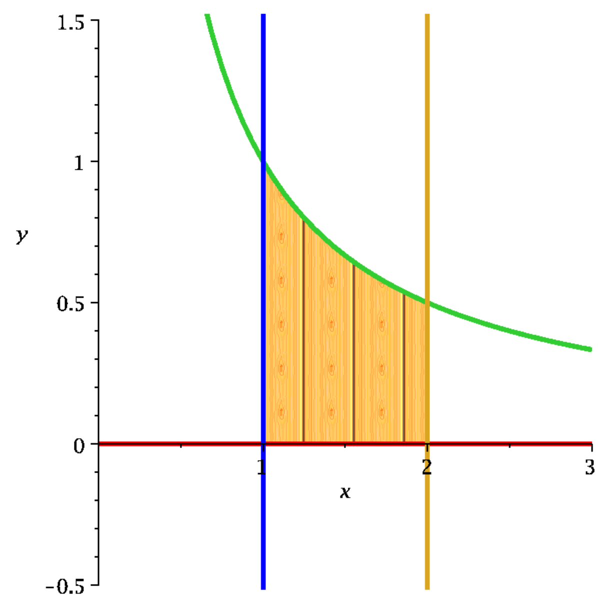
We define the **area** A of S as the limiting value of the sum of the areas of the above approximating rectangles:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\, \Delta x$$

From this it follows that the area A of the region bounded by the curves $y = f(x), y = g(x)$, and the lines $x = a, x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]dx \tag{1}$$

EXAMPLE: Find the area of the region bounded above by $y = \dfrac{1}{x}$, bounded below by the x-axis, and bounded on the sides by $x = 1$ and $x = 2$.



Solution 1: Since $y = \dfrac{1}{x}$ is positive on $[1, 2]$, we have

$$A = \int_1^2 \frac{1}{x}dx = \ln|x|\Big]_1^2 = \ln 2 - \ln 1 = \ln 2 - 0 = \ln 2$$

Solution 2: We can use (1) with $f(x) = \dfrac{1}{x}$ and $g(x) = 0$, since $f(x) > g(x)$ on $[1, 2]$. We have

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]dx = \int_1^2 \left(\frac{1}{x} - 0\right) dx = \int_1^2 \frac{1}{x}dx \\ &= \ln|x|\Big]_1^2 = \ln 2 - \ln 1 = \ln 2 - 0 = \ln 2 \end{aligned}$$

EXAMPLE: Find the area of the region bounded above by the x-axis, bounded below by $y = x^3$, and bounded on the sides by $x = -1$ and $x = 0$.

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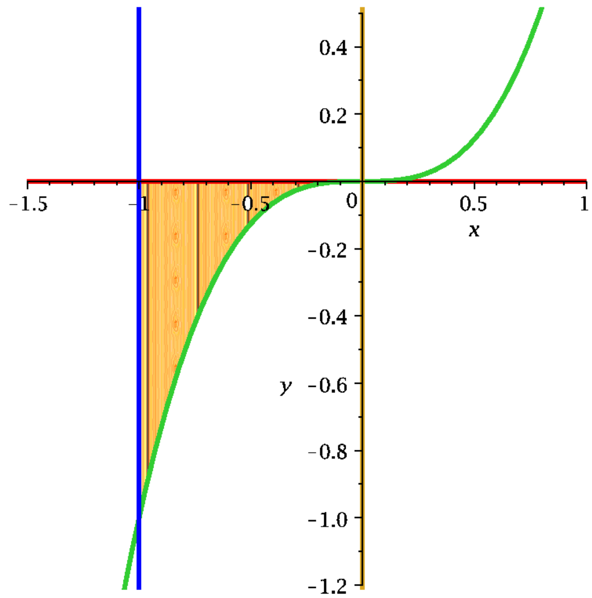


Solution 1: Since $x^3 \le 0$ on $[-1, 0]$, we have

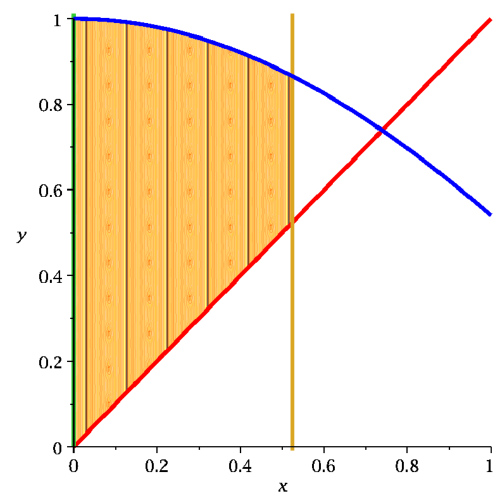
$$A = -\int_{-1}^{0} x^3 dx = -\frac{x^4}{4}\bigg]_{-1}^{0} = -\frac{0^4}{4} + \frac{(-1)^4}{4} = \frac{1}{4}$$

Solution 2: We can use (1) with $f(x) = 0$ and $g(x) = x^3$, since $f(x) \ge g(x)$ on $[-1, 0]$. We have

$$A = \int_a^b [f(x) - g(x)]dx = \int_{-1}^{0} \left(0 - x^3\right) dx = -\int_{-1}^{0} x^3 dx$$

$$= -\frac{x^4}{4}\bigg]_{-1}^{0} = -\frac{0^4}{4} + \frac{(-1)^4}{4} = \frac{1}{4}$$

EXAMPLE: Find the area of the region bounded by $y = x,\ y = \cos x,\ x = 0$, and $x = \pi/6$.



Solution: We can use (1) with $f(x) = \cos x$ and $g(x) = x$, since $f(x) > g(x)$ on $[0, \pi/6]$. We have

$$A = \int_a^b [f(x) - g(x)]dx = \int_0^{\pi/6} (\cos x - x)\, dx$$

$$= \left[\sin x - \frac{x^2}{2}\right]_0^{\pi/6} = \left(\sin\left(\frac{\pi}{6}\right) - \frac{(\pi/6)^2}{2}\right) - \left(\sin 0 - \frac{0^2}{2}\right)$$

$$= \frac{1}{2} - \frac{\pi^2}{72}$$

EXAMPLE: Find the area of the region bounded by $y = 9 - x^2,\ y = x + 1,\ x = -1$, and $x = 2$.

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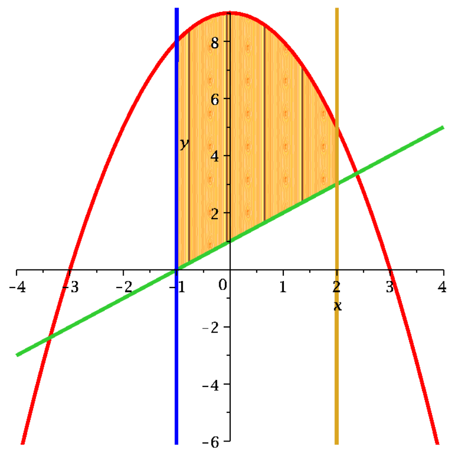
Solution: We can use (1) with $f(x) = 9 - x^2$ and $g(x) = x + 1$, since $f(x) > g(x)$ on $[-1, 2]$. We have



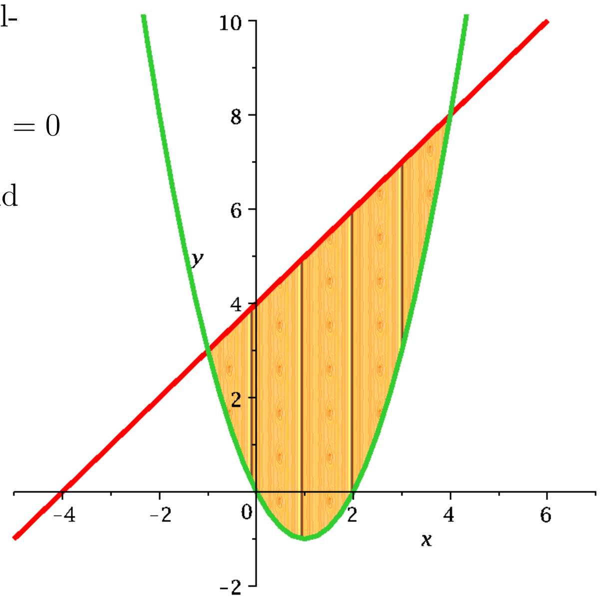
$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]dx = \int_{-1}^{2} \left[(9 - x^2) - (x+1)\right] dx \\ &= \int_{-1}^{2} \left(8 - x - x^2\right) dx = \left[8x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2} \\ &= \left(16 - 2 - \frac{8}{3}\right) - \left(-8 - \frac{1}{2} + \frac{1}{3}\right) = 22 - 3 + \frac{1}{2} = \frac{39}{2} \end{aligned}$$

EXAMPLE: Find the area of the region bounded by $y = x^2 - 2x$ and $y = x + 4$.

Solution: To find the points of intersection we solve the following equation

$$x^2 - 2x = x + 4 \quad \iff \quad x^2 - 3x - 4 = 0 \quad \iff \quad (x+1)(x-4) = 0$$

therefore $x = -1, 4$. We can use (1) with $f(x) = x + 4$ and $g(x) = x^2 - 2x$, since $f(x) \geq g(x)$ on $[-1, 4]$. We have



$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]dx = \int_{-1}^{4} \left[(x+4) - (x^2 - 2x)\right] dx \\ &= \int_{-1}^{4} \left(-x^2 + 3x + 4\right) dx = \left[-\frac{x^3}{3} + \frac{3x^2}{2} + 4x\right]_{-1}^{4} \\ &= \left(-\frac{64}{3} + 24 + 16\right) - \left(\frac{1}{3} + \frac{3}{2} - 4\right) = \frac{125}{6} \end{aligned}$$

EXAMPLE: Find the area of the region bounded by $y = \sqrt{x+3}$ and $y = \dfrac{x+3}{2}$.

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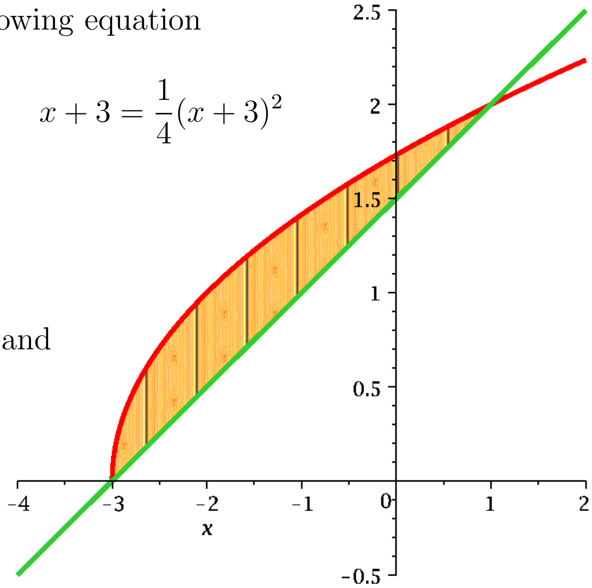

Solution: To find the points of intersection we solve the following equation

$$\sqrt{x+3}=\frac{x+3}{2} \iff \left(\sqrt{x+3}\right)^2=\left(\frac{x+3}{2}\right)^2 \iff x+3=\frac{1}{4}(x+3)^2$$

which can be rewritten as

$$4(x+3)-(x+3)^2=0 \iff (x+3)(1-x)=0$$

therefore $x=-3,1$. We can use (1) with $f(x)=\sqrt{x+3}$ and $g(x)=\dfrac{x+3}{2}$, since $f(x)\geq g(x)$ on $[-3,1]$. We have

$$\begin{aligned} A=\int\limits_a^b [f(x)-g(x)]dx &= \int\limits_{-3}^{1}\left(\sqrt{x+3}-\frac{x+3}{2}\right)dx \\ &= \left[\frac{2}{3}(x+3)^{3/2}-\frac{(x+3)^2}{4}\right]_{-3}^{1} \\ &= \left(\frac{16}{3}-4\right)-(0-0)=\frac{4}{3} \end{aligned}$$

EXAMPLE: Find the area of the region bounded by $y=\sqrt{x}$, $y=\dfrac{x}{2}$, and $x=5$.

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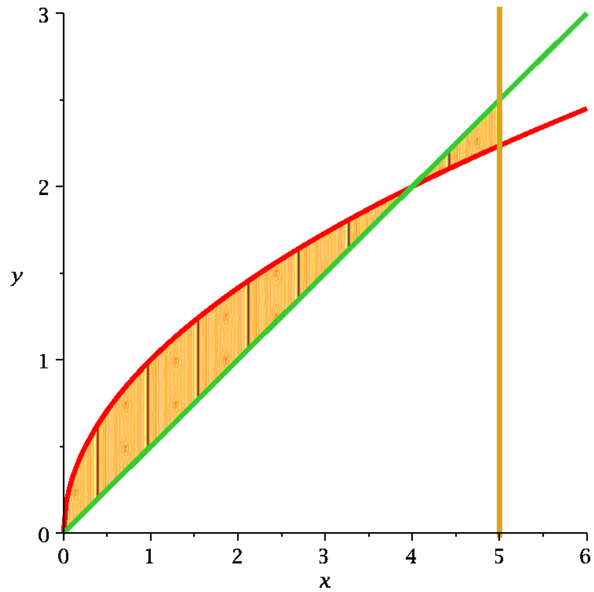
Solution: To find the points of intersection we solve the following equation

$$\sqrt{x}=\frac{x}{2} \iff (\sqrt{x})^2=\left(\frac{x}{2}\right)^2 \iff x=\frac{1}{4}x^2$$

which can be rewritten as

$$x^2-4x=0 \iff x(x-4)=0$$

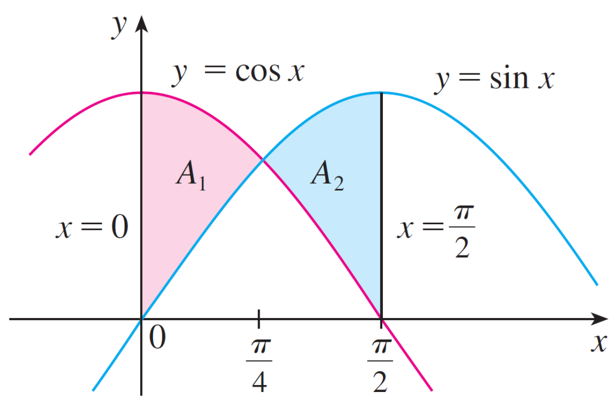
therefore $x=0,4$. Note that $\sqrt{x}\ge\dfrac{x}{2}$ on $[0,4]$ and $\dfrac{x}{2}\ge\sqrt{x}$ on $[4,9]$. Therefore



$$A=\int_0^4\left(\sqrt{x}-\frac{x}{2}\right)dx+\int_4^5\left(\frac{x}{2}-\sqrt{x}\right)dx$$
$$=\left[\frac{2}{3}x^{3/2}-\frac{x^2}{4}\right]_0^4+\left[\frac{x^2}{4}-\frac{2}{3}x^{3/2}\right]_4^5=\frac{107-40\sqrt{5}}{12}$$

EXAMPLE: Find the area of the region bounded by $y=\sin x$, $y=\cos x$, $x=0$, and $x=\pi/2$.

Solution: The points of intersection occur when $\sin x=\cos x$, that is, when $x=\pi/4$ (since $0\le x\le\pi/2$). Observe that $\cos x\ge\sin x$ when $0\le x\le\pi/4$ but $\sin x\ge\cos x$ when $\pi/4\le x\le\pi/2$. Therefore the required area is



$$A=\int_0^{\pi/2}|\cos x-\sin x|dx$$
$$=\int_0^{\pi/4}(\cos x-\sin x)dx+\int_{\pi/4}^{\pi/2}(\sin x-\cos x)dx=[\sin x+\cos x]_0^{\pi/4}+[-\cos x-\sin x]_{\pi/4}^{\pi/2}$$
$$=\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-0-1\right)+\left(-0-1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)=2\sqrt{2}-2$$

EXAMPLE: Find the area enclosed by the line $y=x-1$ and the parabola $y^2=2x+6$.

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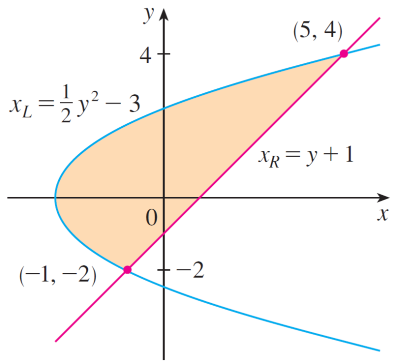

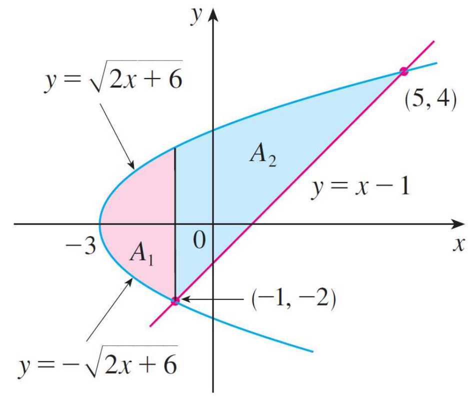

Solution 1: By solving the equation

$$y + 1 = \frac{y^2 - 6}{2}$$

we find that the points of intersection are $(-1, -2)$ and $(5, 4)$. We solve the equation of the parabola for x and notice from the picture that the left and right boundary curves are

$$x_L = \frac{1}{2}y^2 - 3 \quad \text{and} \quad x_R = y + 1$$

We must integrate between the appropriate y-values, $y = -2$ and $y = 4$. Thus

$$\begin{aligned} A &= \int_{-2}^{4} (x_R - x_L) dy = \int_{-2}^{4} \left[(y+1) - \left(\frac{1}{2}y^2 - 3 \right) \right] dy = \int_{-2}^{4} \left(-\frac{1}{2}y^2 + y + 4 \right) dy \\ &= \left[-\frac{1}{2} \cdot \frac{y^3}{3} + \frac{y^2}{2} + 4y \right]_{-2}^{4} = \left(-\frac{1}{6} \cdot 64 + 8 + 16 \right) - \left(\frac{4}{3} + 2 - 8 \right) = 18 \end{aligned}$$

Solution 2: We can find the area by integrating with respect to x instead of y, but the calculation is much more involved:

$$\begin{aligned} A &= \int_{-3}^{-1} [\sqrt{2x+6} - (-\sqrt{2x+6})] dx + \int_{-1}^{5} [\sqrt{2x+6} - (x-1)] dx \\ &= 2\int_{-3}^{-1} \sqrt{2x+6} dx + \int_{-1}^{5} [\sqrt{2x+6} - x + 1] dx \\ &= 2\left[\frac{1}{3}(2x+6)^{3/2} \right]_{-3}^{-1} + \left[\frac{1}{3}(2x+6)^{3/2} - \frac{x^2}{2} + x \right]_{-1}^{5} = 18 \end{aligned}$$