Approximate Integration

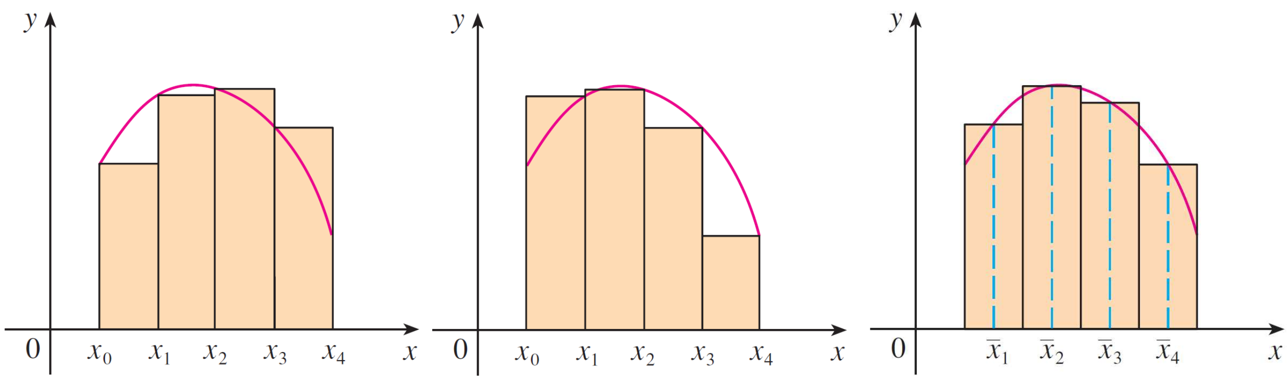
MIDPOINT RULE:

$$\int_a^b f(x)dx \approx M_n = \Delta x[f(\overline{x}_1) + f(\overline{x}_2) + \ldots + f(\overline{x}_n)]$$

where

$$\Delta x = \frac{b-a}{n}$$

and $\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i) =$ midpoint of $[x_{i-1}, x_i]$.



(a) Left endpoint approximation (b) Right endpoint approximation (c) Midpoint approximation

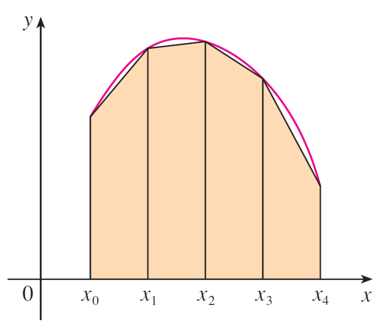
TRAPEZOIDAL RULE:

$$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)]$$

where

$$\Delta x = \frac{b-a}{n}$$

and $x_i = a + i\Delta x$.



EXAMPLE: Use (a) the Trapezoidal Rule and (b) the Midpoint Rule with $n = 5$ to approximate the integral

$$\int_1^2 \frac{1}{x}dx$$

Solution:

(a) With $n = 5, a = 1$, and $b = 2$, we have $\Delta x = (2-1)/5 = 0.2$, and so the Trapezoidal Rule gives

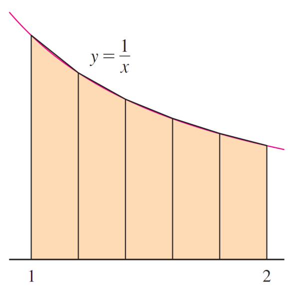
$$\int_1^2 \frac{1}{x}dx \approx T_5 = \frac{\Delta x}{2}[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)]$$

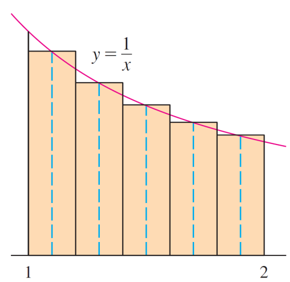
$$= \frac{0.2}{2}\left(\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right) \approx 0.695635$$

(b) The midpoints of the five subintervals are $1.1, 1.3, 1.5, 1.7$, and 1.9, so the Midpoint Rule gives

$$\int_1^2 \frac{1}{x}dx \approx M_5 = \Delta x[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)]$$

$$= \frac{1}{5}\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right) \approx 0.691908$$





REMARK: Note that

$$\int_1^2 \frac{1}{x}dx = \ln x\Big|_1^2 = \ln 2 \approx 0.693147$$

therefore the errors in the Trapezoidal and Midpoint Rule approximations for $n = 5$ are

$$E_T \approx -0.002488 \quad \text{and} \quad E_M \approx 0.001239$$

We see that the size of the error in the Midpoint Rule is about **half the size** of the error in the Trapezoidal Rule.

EXAMPLE: Use (a) the Trapezoidal Rule and (b) the Midpoint Rule with $n = 10$ to approximate the integral

$$\int_0^1 e^{x^2}dx$$

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Solution:

(a) With $n = 10, a = 0$, and $b = 1$, we have $\Delta x = (1 - 0)/10 = 0.1$, and so the Trapezoidal Rule gives

$$\int_0^1 e^{x^2} dx \approx T_{10} = \frac{\Delta x}{2}[f(0) + 2f(0.1) + 2f(0.2) + \ldots + 2f(0.8) + 2f(0.9) + f(1)]$$

$$= \frac{0.1}{2}\left(e^{0^2} + 2e^{0.1^2} + 2e^{0.2^2} + \ldots + 2e^{0.8^2} + 2e^{0.9^2} + e^{1^2}\right) \approx 1.467174693$$

(b) The midpoints of the ten subintervals are $0.05, 0.15, 0.25, \ldots, 0.85, 0.95$, so the Midpoint Rule gives

$$\int_0^1 e^{x^2} dx \approx M_{10} = \Delta x[f(0.05) + f(0.15) + f(0.25) + \ldots + f(0.85) + f(0.95)]$$

$$= \frac{1}{10}\left(e^{0.05^2} + e^{0.15^2} + e^{0.25^2} + \ldots + e^{0.85^2} + e^{0.95^2}\right) \approx 1.460393091$$

REMARK: One can compute that

$$\int_0^1 e^{x^2} dx \approx 1.462651746$$

therefore the errors in the Trapezoidal and Midpoint Rule approximations for $n = 10$ are

$$E_T \approx -0.004522947 \quad \text{and} \quad E_M \approx 0.002258655$$

We see that the size of the error in the Midpoint Rule is about **half the size** of the error in the Trapezoidal Rule.

EXAMPLE: Use (a) the Trapezoidal Rule and (b) the Midpoint Rule with $n = 10$ to approximate the integral

$$\int_0^1 \sqrt{1 + x^3} dx$$

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$$\int_0^1 \sqrt{1+x^3}dx$$

Solution:

(a) With $n=10, a=0$, and $b=1$, we have $\Delta x=(1-0)/10=0.1$, and so the Trapezoidal Rule gives

$$\int_0^1 \sqrt{1+x^3}dx \approx T_{10}=\frac{\Delta x}{2}[f(0)+2f(0.1)+2f(0.2)+\ldots+2f(0.8)+2f(0.9)+f(1)]$$

$$=\frac{0.1}{2}\left(\sqrt{1+0^3}+2\sqrt{1+0.1^3}+2\sqrt{1+0.2^3}+\ldots+2\sqrt{1+0.8^3}+2\sqrt{1+0.9^3}+\sqrt{1+1^3}\right)$$

$$\approx 1.112332391$$

(b) The midpoints of the ten subintervals are $0.05, 0.15, 0.25, \ldots, 0.85, 0.95$ so the Midpoint Rule gives

$$\int_0^1 \sqrt{1+x^3}dx \approx M_{10}=\Delta x[f(0.05)+f(0.15)+f(0.25)+\ldots+f(0.85)+f(0.95)]$$

$$=\frac{1}{10}\left(\sqrt{1+0.05^3}+\sqrt{1+0.15^3}+\sqrt{1+0.25^3}+\ldots+\sqrt{1+0.85^3}+\sqrt{1+0.95^3}\right)$$

$$\approx 1.111005559$$

REMARK: One can compute that

$$\int_0^1 \sqrt{1+x^3}dx \approx 1.111447979$$

therefore the errors in the Trapezoidal and Midpoint Rule approximations for $n=10$ are

$$E_T \approx -0.000884412 \quad \text{and} \quad E_M \approx 0.000442420$$

We see that the size of the error in the Midpoint Rule is about **half the size** of the error in the Trapezoidal Rule.

ERROR BOUNDS: Suppose $|f''(x)| \le K$ for $a \le x \le b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then

$$\boxed{|E_T| \le \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_M| \le \frac{K(b-a)^3}{24n^2}}$$

EXAMPLE: Give upper bounds for the errors E_T and E_M involved in the approximation of $\int_1^2 \frac{1}{x}dx$ with $n=5$.

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Solution: Note that

$$f'(x)=-\frac{1}{x^2} \quad \text{and} \quad f''(x)=\frac{2}{x^3}$$

Since $\frac{2}{x^3}$ is a decreasing function on $[1,2]$, we have

$$|f''(x)|=\left|\frac{2}{x^3}\right| \le \frac{2}{1^3}=2$$

Therefore, taking $K=2, a=1, b=2$, and $n=5$ in the above error estimates, we obtain

$$|E_T| \le \frac{K(b-a)^3}{12n^2}=\frac{2(2-1)^3}{12(5)^2}=\frac{1}{150}\approx 0.0067$$

and

$$|E_M| \le \frac{K(b-a)^3}{24n^2}=\frac{2(2-1)^3}{24(5)^2}=\frac{1}{300}\approx 0.0033$$

REMARK: Note that these error estimates are bigger than the actual errors 0.002488 and 0.001239.

EXAMPLE: Give upper bounds for the errors E_T and E_M involved in the approximation of $\int\limits_0^1 e^{x^2}dx$ with $n=10$.

Solution: Note that

$$f'(x)=2xe^{x^2} \quad \text{and} \quad f''(x)=2e^{x^2}+4x^2e^{x^2}$$

Since $2e^{x^2}+4x^2e^{x^2}$ is an increasing function on $[0,1]$, we have

$$|f''(x)|=2e^{x^2}+4x^2e^{x^2} \le 2e^{1^2}+4(1)^2e^{1^2}=6e$$

Therefore, taking $K=6e, a=0, b=1$, and $n=10$ in the above error estimates, we obtain

$$|E_T| \le \frac{K(b-a)^3}{12n^2}=\frac{6e(1-0)^3}{12(10)^2}=\frac{e}{200}\approx 0.01359140914$$

and

$$|E_M| \le \frac{K(b-a)^3}{24n^2}=\frac{6e(1-0)^3}{24(10)^2}=\frac{e}{400}\approx 0.006795704570$$

REMARK: Note that these error estimates are bigger than the actual errors 0.004522947 and 0.002258655.

EXAMPLE: Give upper bounds for the errors E_T and E_M involved in the approximation of $\int\limits_0^1 \sqrt{1+x^3}dx$ with $n=10$.

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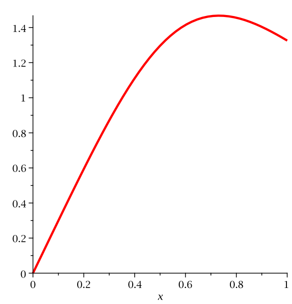
Solution: Note that (see the Appendix)

$$f'(x) = \frac{3x^2}{2\sqrt{1+x^3}} \quad \text{and} \quad f''(x) = \frac{3x(x^3+4)}{4(1+x^3)^{3/2}}$$

We now find increasing/decreasing intervals of $f''(x)$. Here is the graph of $f''(x)$:



To find a point of a local maximum, we note that (see the Appendix)

$$f'''(x) = -\frac{3(x^6+20x^3-8)}{8(1+x^3)^{5/2}}$$

One can check that $f'''(x) = 0$ on $[0, 1]$ at $x \approx 0.7320508076$ which is a root of $x^6+20x^3-8 = 0$. It is easy to show that this is a point of a local maximum of $f''(x)$. So,

$$|f''(x)| = \left|\frac{3x(x^3+4)}{4(1+x^3)^{3/2}}\right| \le \left.\frac{3x(x^3+4)}{4(1+x^3)^{3/2}}\right|_{x=0.7320508076\ldots} \approx 1.467889825$$

Therefore, taking $K = 1.467889825, a = 0, b = 1$, and $n = 10$ in the above error estimates, we obtain

$$|E_T| \le \frac{K(b-a)^3}{12n^2} = \frac{1.467889825(1-0)^3}{12(10)^2} \approx 0.001223241521$$

and

$$|E_M| \le \frac{K(b-a)^3}{24n^2} = \frac{1.467889825(1-0)^3}{24(10)^2} \approx 0.0006116207604$$

REMARK: Note that these error estimates are bigger than the actual errors 0.000884412 and 0.000442420.

EXAMPLE: How large should we take n in order to guarantee that the Trapezoidal and Midpoint Rule approximations for $\int\limits_1^2 \frac{1}{x}dx$ are accurate to within 0.0001?

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Solution: We saw in one of the previous examples that $|f''(x)| \leq 2$ for $1 \leq x \leq 2$, so we can take $K = 2, a = 1$, and $b = 2$ in

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_M| \leq \frac{K(b-a)^3}{24n^2}$$

Accuracy to within 0.0001 means that the size of the error should be less than 0.0001. Therefore, we choose n so that

$$\frac{2 \cdot 1^3}{12n^2} < 0.0001 \quad \text{(Trapezoidal Rule)}$$

Solving the inequality for n, we get

$$n^2 > \frac{2}{12(0.0001)} \quad \Longrightarrow \quad n > \frac{1}{\sqrt{0.0006}} \approx 40.8$$

Thus $n = 41$ will ensure the desired accuracy.

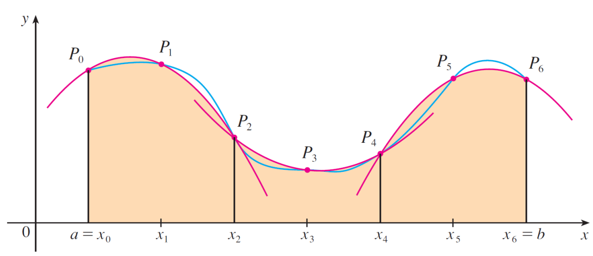
For the same accuracy with the Midpoint Rule we choose n so that

$$\frac{2 \cdot 1^3}{24n^2} < 0.0001 \quad \Longrightarrow \quad n > \frac{1}{\sqrt{0.0012}} \approx 29$$

SIMPSON'S RULE:

$$\int\limits_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

where n is even and $\Delta x = \frac{b-a}{n}$.



ERROR BOUND FOR SIMPSON'S RULE: Suppose that $|f^{(4)}(x)| \leq K$ for $a \leq x \leq b$. If E_S is the error involved in using Simpson's Rule, then

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

EXAMPLE: How large should we take n in order to guarantee that the Simpson's Rule approximation for $\int\limits_0^2 x^3 dx$ is accurate to within 0.0001?

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Solution: Note that

$$f'(x)=3x^2,\quad f''(x)=6x,\quad f'''(x)=6,\quad \text{and}\quad f^{(4)}(x)=0$$

Therefore, taking $K=0$ in the above error estimate, we obtain

$$|E_S|\leq\frac{K(b-a)^5}{180n^4}=\frac{0\cdot(b-a)^5}{180n^4}=0$$

This means that Simpson's Rule gives the *exact* value of $\int\limits_0^2 x^3dx$ with $n=2$. In fact,

$$\int\limits_0^2 x^3dx=\left.\frac{x^4}{4}\right|_0^2=\frac{2^4}{4}=4$$

which is the same as

$$\frac{1}{3}[0^3+4\cdot1^3+2^3]$$

REMARK: One can show that if f is a polynomial of degree 3 or lower, then Simpson's Rule gives the exact value of $\int\limits_a^b f(x)dx$.

EXAMPLE: How large should we take n in order to guarantee that the Simpson's Rule approximation for $\int\limits_1^2 \frac{1}{x}dx$ is accurate to within 0.0001?

Solution: If $f(x)=1/x$, then $f^{(4)}(x)=24/x^5$. Since $24/x^5$ is a decreasing function on $[1,2]$, we have

$$|f^{(4)}(x)|=\left|\frac{24}{x^5}\right|\leq\frac{24}{1^5}=24$$

Therefore, we can take $K=24,\ a=1$, and $b=2$ in

$$|E_S|\leq\frac{K(b-a)^5}{180n^4}$$

Accuracy to within 0.0001 means that the size of the error should be less than 0.0001. Therefore, we choose n so that

$$\frac{24\cdot1^5}{180n^4}<0.0001$$

Solving the inequality for n, we get

$$n^4>\frac{24}{180(0.0001)}\quad\Longrightarrow\quad n>\frac{1}{\sqrt[4]{0.00075}}\approx6.04$$

Thus $n=8$ (n must be even) will ensure the desired accuracy.

Appendix

EXAMPLE: Let $f(x) = \sqrt{1+x^3}$. Find f', f'', and f'''.

Solution: Since $f(x) = (1+x^3)^{1/2}$, we have

$$f'(x) = \frac{1}{2}(1+x^3)^{1/2-1} \cdot (1+x^3)' = \frac{1}{2}(1+x^3)^{-1/2} \cdot 3x^2 = \boxed{\frac{3x^2}{2\sqrt{1+x^3}}}$$

$$\begin{aligned}
f''(x) = \left(\frac{3x^2}{2(1+x^3)^{1/2}}\right)' &= \frac{3}{2}\left(\frac{x^2}{(1+x^3)^{1/2}}\right)' \\
&= \frac{3}{2} \cdot \frac{(x^2)'(1+x^3)^{1/2} - x^2[(1+x^3)^{1/2}]'}{[(1+x^3)^{1/2}]^2} \\
&= \frac{3}{2} \cdot \frac{2x(1+x^3)^{1/2} - x^2\frac{1}{2}(1+x^3)^{1/2-1} \cdot (1+x^3)'}{1+x^3} \\
&= \frac{3}{2} \cdot \frac{2x(1+x^3)^{1/2} - x^2\frac{1}{2}(1+x^3)^{-1/2} \cdot 3x^2}{1+x^3} \\
&= \frac{3}{2} \cdot \frac{2x(1+x^3)^{1/2} - \frac{3}{2}x^4(1+x^3)^{-1/2}}{1+x^3} \\
&= \frac{3}{2} \cdot \frac{\left(2x(1+x^3)^{1/2} - \frac{3}{2}x^4(1+x^3)^{-1/2}\right) \cdot 2(1+x^3)^{1/2}}{(1+x^3) \cdot 2(1+x^3)^{1/2}} \\
&= \frac{3}{2} \cdot \frac{2x(1+x^3)^{1/2} \cdot 2(1+x^3)^{1/2} - \frac{3}{2}x^4(1+x^3)^{-1/2} \cdot 2(1+x^3)^{1/2}}{2(1+x^3)^{3/2}} \\
&= \frac{3}{2} \cdot \frac{4x(1+x^3) - 3x^4}{2(1+x^3)^{3/2}} \\
&= \frac{3}{2} \cdot \frac{4x + 4x^4 - 3x^4}{2(1+x^3)^{3/2}} \\
&= \frac{3}{2} \cdot \frac{4x + x^4}{2(1+x^3)^{3/2}} \\
&= \frac{3}{2} \cdot \frac{x(4+x^3)}{2(1+x^3)^{3/2}} \\
&= \boxed{\frac{3x(4+x^3)}{4(1+x^3)^{3/2}}}
\end{aligned}$$

$$f'''(x) = \left(\frac{3x(4+x^3)}{4(1+x^3)^{3/2}}\right)'$$

$$= \frac{3}{4}\left(\frac{x(4+x^3)}{(1+x^3)^{3/2}}\right)'$$

$$= \frac{3}{4}\cdot\frac{[x(4+x^3)]'(1+x^3)^{3/2} - x(4+x^3)[(1+x^3)^{3/2}]'}{[(1+x^3)^{3/2}]^2}$$

$$= \frac{3}{4}\cdot\frac{[x'(4+x^3)+x(4+x^3)'](1+x^3)^{3/2} - x(4+x^3)\frac{3}{2}(1+x^3)^{3/2-1}\cdot(1+x^3)'}{(1+x^3)^3}$$

$$= \frac{3}{4}\cdot\frac{[1\cdot(4+x^3)+x\cdot 3x^2](1+x^3)^{3/2} - x(4+x^3)\frac{3}{2}(1+x^3)^{1/2}\cdot 3x^2}{(1+x^3)^3}$$

$$= \frac{3}{4}\cdot\frac{(4+x^3+3x^3)(1+x^3)^{3/2} - \frac{9}{2}x^3(4+x^3)(1+x^3)^{1/2}}{(1+x^3)^3}$$

$$= \frac{3}{4}\cdot\frac{(4+4x^3)(1+x^3)^{3/2} - \frac{9}{2}x^3(4+x^3)(1+x^3)^{1/2}}{(1+x^3)^3}$$

$$= \frac{3}{4}\cdot\frac{4(1+x^3)(1+x^3)^{3/2} - \frac{9}{2}x^3(4+x^3)(1+x^3)^{1/2}}{(1+x^3)^3}$$

$$= \frac{3}{4}\cdot\frac{4(1+x^3)^{5/2} - \frac{9}{2}x^3(4+x^3)(1+x^3)^{1/2}}{(1+x^3)^3}$$

$$= \frac{3}{4}\cdot\frac{\left(4(1+x^3)^{5/2} - \frac{9}{2}x^3(4+x^3)(1+x^3)^{1/2}\right)\cdot 2(1+x^3)^{-1/2}}{(1+x^3)^3\cdot 2(1+x^3)^{-1/2}}$$

$$= \frac{3}{4}\cdot\frac{4(1+x^3)^{5/2}\cdot 2(1+x^3)^{-1/2} - \frac{9}{2}x^3(4+x^3)(1+x^3)^{1/2}\cdot 2(1+x^3)^{-1/2}}{2(1+x^3)^{5/2}}$$

$$= \frac{3}{4}\cdot\frac{8(1+x^3)^2 - 9x^3(4+x^3)}{2(1+x^3)^{5/2}} = \frac{3}{4}\cdot\frac{8(1+2x^3+x^6) - 36x^3 - 9x^6}{2(1+x^3)^{5/2}}$$

$$= \frac{3}{4}\cdot\frac{8+16x^3+8x^6-36x^3-9x^6}{2(1+x^3)^{5/2}} = \frac{3}{4}\cdot\frac{8-20x^3-x^6}{2(1+x^3)^{5/2}}$$

$$= \frac{3(8-20x^3-x^6)}{8(1+x^3)^{5/2}} = \boxed{-\frac{3(x^6+20x^3-8)}{8(1+x^3)^{5/2}}}$$