# The Substitution Rule

THEOREM (The Fundamental Theorem Of Calculus, Part II): If $f$ is continuous on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big]_a^b$$

where $F$ is any antiderivative of $f$, that is $F' = f$.

NOTATION: To denote the set of all antiderivatives of $f$ on an (open) interval $I$ we use the **indefinite integral** notation:

$$\int f(x)dx = F(x) + C$$

### Table Of Indefinite Integrals

| | |
|---|---|
| $\int cf(x)dx = c\int f(x)dx$ | $\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$ |
| $\int x^n dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$ | $\int \frac{1}{x}dx = \ln\lvert x\rvert + C$ |
| $\int e^x dx = e^x + C$ | $\int a^x dx = \frac{a^x}{\ln a} + C$ |
| $\int \sin x dx = -\cos x + C$ | $\int \cos x dx = \sin x + C$ |
| $\int \sec^2 x dx = \tan x + C$ | $\int \csc^2 x dx = -\cot x + C$ |
| $\int \sec x \tan x dx = \sec x + C$ | $\int \csc x \cot x dx = -\csc x + C$ |
| $\int \frac{1}{1+x^2}dx = \arctan x + C$ | $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$ |

EXAMPLES:

1. $\int x^2 dx = [PR\ with\ n=2] = \frac{x^{2+1}}{2+1} + C = \frac{x^3}{3} + C$

2. $\int \frac{1}{\sqrt[5]{x}}dx = \int \frac{1}{x^{1/5}}dx = \int x^{-1/5}dx = [PR\ with\ n=-1/5] = \frac{x^{-1/5+1}}{-1/5+1} + C = \frac{5}{4}x^{4/5} + C$

3. $\int x\sqrt[3]{x}dx = \int x^1 \cdot x^{1/3}dx = \int \underbrace{x^{1+1/3}}_{x^{4/3}}dx = [PR\ with\ n=4/3] = \frac{x^{4/3+1}}{4/3+1} + C = \frac{3}{7}x^{7/3} + C$

4. $\int \frac{x\sqrt[3]{x}}{\sqrt{x}}dx = \int \frac{x^1 \cdot x^{1/3}}{x^{1/2}}dx = \int \frac{x^{1+1/3}}{x^{1/2}}dx = \int \underbrace{x^{1+1/3-1/2}}_{x^{5/6}}dx = [PR\ with\ n=5/6] = \frac{x^{5/6+1}}{5/6+1} + C$

$$= \frac{6}{11}x^{11/6} + C$$

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5. $$\int \left(\frac{4-5\sqrt{x}+7x\sin x}{9x}\right)dx = \int\left(\frac{4}{9x}-\frac{5x^{1/2}}{9x}+\frac{7x\sin x}{9x}\right)dx = \int\left(\frac{4}{9}\cdot\frac{1}{x}-\frac{5}{9}x^{-1/2}+\frac{7}{9}\sin x\right)dx$$

$$= \int \frac{4}{9}\cdot\frac{1}{x}dx - \int\frac{5}{9}x^{-1/2}dx + \int\frac{7}{9}\sin x dx \quad = \frac{4}{9}\int\frac{1}{x}dx - \frac{5}{9}\int x^{-1/2}dx + \frac{7}{9}\int \sin x dx$$

$$= \frac{4}{9}\ln\lvert x\rvert - \frac{5}{9}\cdot\frac{x^{-1/2+1}}{-1/2+1} - \frac{7}{9}\cos x + C$$

$$= \frac{4}{9}\ln\lvert x\rvert - \frac{10}{9}x^{1/2} - \frac{7}{9}\cos x + C$$

6. $$\int_0^1 \frac{x^3+x+1}{x^2+1}dx = \int_0^1\left(\frac{x^3+x}{x^2+1}+\frac{1}{x^2+1}\right)dx = \int_0^1\left(\frac{x(x^2+1)}{x^2+1}+\frac{1}{x^2+1}\right)dx = \int_0^1\left(x+\frac{1}{x^2+1}\right)dx$$

$$= \left[\frac{x^2}{2}+\arctan x\right]_0^1 = \left(\frac{1^2}{2}+\arctan 1\right) - \left(\frac{0^2}{2}+\arctan 0\right) = \frac{1}{2}+\arctan 1 = \frac{1}{2}+\frac{\pi}{4}$$

7. $$\int x(1+x^2)^2dx = \int x\left(1+2x^2+x^4\right)dx = \int (x+2x^3+x^5)dx \quad = \frac{x^2}{2}+2\frac{x^4}{4}+\frac{x^6}{6}+C$$

$$= \frac{1}{2}x^2+\frac{1}{2}x^4+\frac{1}{6}x^6+C$$

8. $$\int x(1+x^2)^3dx = \int x\left(1+3x^2+3x^4+x^6\right)dx = \int (x+3x^3+3x^5+x^7)dx$$

$$= \frac{x^2}{2}+3\frac{x^4}{4}+3\frac{x^6}{6}+\frac{x^8}{8}+C = \frac{1}{2}x^2+\frac{3}{4}x^4+\frac{1}{2}x^6+\frac{1}{8}x^8+C$$

9. $\int x(1+x^2)^{50}dx = ???$

ANSWER 1:

$$\int x(1+x^2)^{50}dx$$
$$= (x^{102} + 51\,x^{100} + 1275\,x^{98} + 20825\,x^{96} + 249900\,x^{94} + 2349060\,x^{92} + 18009460\,x^{90}$$
$$+115775100\,x^{88} + 636763050\,x^{86} + 3042312350\,x^{84} + 12777711870\,x^{82}$$
$$+47626016970\,x^{80} + 158753389900\,x^{78} + 476260169700\,x^{76} + 1292706174900\,x^{74}$$
$$+3188675231420\,x^{72} + 7174519270695\,x^{70} + 14771069086725\,x^{68} + 27900908274925\,x^{66}$$
$$+48459472266975\,x^{64} + 77535155627160\,x^{62} + 114456658306760\,x^{60} + 156077261327400\,x^{58}$$
$$+196793068630200\,x^{56} + 229591913401900\,x^{54} + 247959266474052\,x^{52} + 247959266474052\,x^{50}$$
$$+229591913401900\,x^{48} + 196793068630200\,x^{46} + 156077261327400\,x^{44} + 114456658306760\,x^{42}$$
$$+77535155627160\,x^{40} + 48459472266975\,x^{38} + 27900908274925\,x^{36} + 14771069086725\,x^{34}$$
$$+7174519270695\,x^{32} + 3188675231420\,x^{30} + 1292706174900\,x^{28} + 476260169700\,x^{26}$$
$$+158753389900\,x^{24} + 47626016970\,x^{22} + 12777711870\,x^{20} + 3042312350\,x^{18}$$
$$+636763050\,x^{16} + 115775100\,x^{14} + 18009460\,x^{12} + 2349060\,x^{10}$$
$$+249900\,x^{8} + 20825\,x^{6} + 1275\,x^{4} + 51\,x^{2})/102 + C$$

ANSWER 2:

$$\int x(1+x^2)^{50}dx = \frac{1}{102}(1+x^2)^{51} + C$$

because

$$\left(\frac{1}{102}(1+x^2)^{51} + C\right)' = \left(\frac{1}{102}(1+x^2)^{51}\right)' + C' = \begin{bmatrix} (cf)' = cf' \\ C' = 0 \end{bmatrix} = \frac{1}{102}\left((1+x^2)^{51}\right)'$$

$$= [(u^n)' = nu^{n-1}\cdot u'] = \frac{1}{102}\cdot 51(1+x^2)^{50}(1+x^2)' = \frac{1}{102}\cdot 51(1+x^2)^{50}\cdot 2x = x(1+x^2)^{50}$$

THEOREM (The Substitution Rule): If $u = g(x)$ is a differentiable function whose range is an interval $I$ and $f$ is continuous on $I$, then

$$\boxed{\int f(g(x))g'(x)dx = \int f(u)du}$$

SOLUTION:

$$\int \underline{\underline{x}}(\underline{1+x^2})^{50}\underline{\underline{dx}} = \begin{bmatrix} \underline{1+x^2} = u \\ d(1+x^2) = du \\ 2xdx = du \\ \underline{\underline{xdx}} = \frac{1}{2}du \end{bmatrix} = \int u^{50}\frac{1}{2}du = \frac{1}{2}\int u^{50}du = \frac{1}{2}\cdot\frac{u^{51}}{51} + C = \frac{1}{2}\cdot\frac{(1+x^2)^{51}}{51} + C$$

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10. $$\int 4x^2 \sin(1-5x^3)dx = \begin{bmatrix} 1-5x^3 = u \\ d(1-5x^3) = du \\ -15x^2 dx = du \\ 4x^2 dx = -\frac{4}{15}du \end{bmatrix} = \int \sin u \left(-\frac{4}{15}\right) du = -\frac{4}{15}\int \sin u du$$

$$= -\frac{4}{15}(-\cos u) + C = \frac{4}{15}\cos(1-5x^3) + C$$

11. $$\int \frac{2x+3}{\sqrt{x^2+3x-1}}dx = \int \frac{1}{\sqrt{x^2+3x-1}} \cdot (2x+3)dx = \begin{bmatrix} x^2+3x-1 = u \\ d(x^2+3x-1) = du \\ (2x+3)dx = du \end{bmatrix} = \int \frac{1}{\sqrt{u}}du$$

$$= \int u^{-1/2}du = \frac{u^{-1/2+1}}{-1/2+1} + C = \frac{(x^2+3x-1)^{-1/2+1}}{-1/2+1} + C = 2(x^2+3x-1)^{1/2} + C$$

12. $$\int \frac{\ln x}{x}dx = \int \ln x \cdot \frac{1}{x}dx = \begin{bmatrix} \ln x = u \\ d(\ln x) = du \\ \frac{1}{x}dx = du \end{bmatrix} = \int u du = \frac{u^2}{2} + C = \frac{\ln^2 x}{2} + C$$

13. $$\int x e^{x^2+1} dx$$

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13. $$\int xe^{x^2+1}dx = \begin{bmatrix} x^2+1=u \\ d(x^2+1)=du \\ 2xdx = du \\ xdx = \frac{1}{2}du \end{bmatrix} = \int e^u \frac{1}{2}du = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2+1} + C$$

14. $$\int x\sqrt{x+1}dx = \begin{bmatrix} x+1=u \\ d(x+1)=du \\ dx = du \end{bmatrix} = \int (u-1)\sqrt{u}du = \int (u^{3/2} - u^{1/2})du$$

$$= \frac{u^{3/2+1}}{3/2+1} - \frac{u^{1/2+1}}{1/2+1} + C = \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$$

15. $$\int \frac{x+5}{2x+3}dx = \begin{bmatrix} 2x+3=u \\ d(2x+3)=du \\ 2dx = du \\ dx = \frac{1}{2}du \end{bmatrix} = \int \frac{\frac{u-3}{2}+5}{u}\left(\frac{1}{2}\right)du = \int \frac{u-3+10}{4u}du = \int \frac{u+7}{4u}du$$

$$= \int \frac{u}{4u}du + \int \frac{7}{4u}du = \int \frac{1}{4}du + \frac{7}{4}\int \frac{1}{u}du = \frac{1}{4}u + \frac{7}{4}\ln|u| + C = \frac{1}{4}(2x+3) + \frac{7}{4}\ln|2x+3| + C$$

16. (a) $\int \frac{2x+1}{x^2+x+1}dx$ (b) $\int \frac{x+1}{x^2+x+1}dx$

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| $\int \sec u \tan u du = \sec u + C$ | $\int \csc u \cot u du = -\csc u + C$ |
| $\int \frac{1}{1+u^2}du = \arctan u + C$ | $\int \frac{1}{\sqrt{1-u^2}}du = \arcsin u + C$ |

16(a). $$\int \frac{2x+1}{x^2+x+1}dx = \begin{bmatrix} x^2+x+1 = u \\ d(x^2+x+1) = du \\ (2x+1)dx = du \end{bmatrix} = \int \frac{1}{u}du = \ln|u| + C = \ln(x^2+x+1) + C$$

16(b). $$\int \frac{x+1}{x^2+x+1}dx = \int \frac{x+1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx = \begin{bmatrix} x+\frac{1}{2} = u \\ d\left(x+\frac{1}{2}\right) = du \\ dx = du \end{bmatrix} = \int \frac{u-\frac{1}{2}+1}{u^2+\frac{3}{4}}du$$

$$= \int \frac{u+\frac{1}{2}}{u^2+\frac{3}{4}}du = \int \frac{u}{u^2+\frac{3}{4}}du + \frac{1}{2}\int \frac{1}{u^2+\left(\frac{\sqrt{3}}{2}\right)^2}du = \frac{1}{2}\ln\left(u^2+\frac{3}{4}\right) + \frac{1}{\sqrt{3}}\arctan\frac{2u}{\sqrt{3}} + C$$

$$= \frac{1}{2}\ln(x^2+x+1) + \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + C$$ (For more details, see Appendix I)

17. $$\int \frac{dx}{1-x^2} = \int \frac{dx}{(1-x)(1+x)} = \frac{1}{2}\int\left(\frac{1}{1-x}+\frac{1}{1+x}\right)dx = \frac{1}{2}\int \frac{1}{1-x}dx + \frac{1}{2}\int \frac{1}{1+x}dx$$

$$= -\frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x| + C = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$$ (For more details, see Appendix II)

18. $$\int \frac{1}{2+x^2}dx = \int \frac{1}{2\left(1+\frac{x^2}{2}\right)}dx = \frac{1}{2}\int \frac{1}{1+\frac{x^2}{2}}dx = \frac{1}{2}\int \frac{1}{1+\left(\frac{x}{\sqrt{2}}\right)^2}dx = \begin{bmatrix} x/\sqrt{2} = u \\ d(x/\sqrt{2}) = du \\ dx/\sqrt{2} = du \\ dx = \sqrt{2}du \end{bmatrix}$$

$$= \frac{1}{2}\int \frac{1}{1+u^2}\sqrt{2}du = \frac{\sqrt{2}}{2}\int \frac{1}{1+u^2}du = \frac{1}{\sqrt{2}}\arctan u + C = \frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}} + C$$

19. $$\int e^{-x}dx$$

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| $\int \sec u \tan u du = \sec u + C$ | $\int \csc u \cot u du = -\csc u + C$ |
| $\int \frac{1}{a^2+u^2}du = \frac{1}{a}\arctan\frac{u}{a} + C$ | $\int \frac{1}{\sqrt{a^2-u^2}}du = \arcsin\frac{u}{a} + C$ |

19. $$\int e^{-x}dx = \begin{bmatrix} -x = u \\ d(-x) = du \\ -dx = du \\ dx = -du \end{bmatrix} = \int e^u(-du) = -\int e^u du = -e^u + C = -e^{-x} + C$$

20. $$\int \sin(2-5x)dx = \begin{bmatrix} 2-5x = u \\ d(2-5x) = du \\ -5dx = du \\ dx = -\frac{1}{5}du \end{bmatrix} = \int \sin u\left(-\frac{1}{5}\right)du = -\frac{1}{5}\int \sin u du$$

$$= -\frac{1}{5}(-\cos u) + C = \frac{1}{5}\cos(2-5x) + C$$

21. $$\int \sin(2x)dx$$

$$\int \sin(2x)dx = \int 2\sin x\cos x dx = \begin{bmatrix} \sin x = u \\ d(\sin x) = du \\ \cos x dx = du \end{bmatrix} = \int 2u du$$

$$= u^2 + C = \sin^2 x + C$$

$$\int \sin(2x)dx = \begin{bmatrix} 2x = u \\ d(2x) = du \\ 2dx = du \\ dx = \frac{1}{2}du \end{bmatrix} = \int \sin u \left(\frac{1}{2}\right) du$$

$$= \frac{1}{2}\int \sin u du = \frac{1}{2}(-\cos u) + C = -\frac{1}{2}\cos(2x) + C$$

THEOREM (The Substitution Rule for Definite Integrals): If $g'$ is continuous on $[a, b]$ and $f$ is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

22. Find $\displaystyle\int_0^{\pi/4} \sin(2x)dx$.

**INCORRECT!!!**:

$$\int_0^{\pi/4} \sin(2x)dx = \int_0^{\pi/4} 2\sin x\cos x dx = \begin{bmatrix} \sin x = u \\ d(\sin x) = du \\ \cos x dx = du \end{bmatrix} = \int_0^{\pi/4} 2u du$$

$$= u^2\Big]_0^{\pi/4} = \sin^2 x\Big]_0^{\pi/4} = \sin^2\left(\frac{\pi}{4}\right) - \sin^2 0 = \left(\frac{\sqrt{2}}{2}\right)^2 - 0^2 = \frac{1}{2}$$

METHOD 1:

$$\int \sin(2x)dx = \int 2\sin x\cos x dx = \begin{bmatrix} \sin x = u \\ d(\sin x) = du \\ \cos x dx = du \end{bmatrix} = \int 2u du = u^2 + C = \sin^2 x + C$$

Therefore $\displaystyle\int_0^{\pi/4} \sin(2x)dx = \sin^2 x\Big]_0^{\pi/4} = \sin^2\left(\frac{\pi}{4}\right) - \sin^2 0 = \left(\frac{\sqrt{2}}{2}\right)^2 - 0^2 = \frac{1}{2}$.

METHOD 2:

$$\int_0^{\pi/4} \sin(2x)dx = \int_0^{\pi/4} 2\sin x\cos x dx = \begin{bmatrix} \sin x = u \\ d(\sin x) = du \\ \cos x dx = du \end{bmatrix} = \int_{\sin 0}^{\sin(\pi/4)} 2u du = u^2\Big]_0^{\sqrt{2}/2} = \left(\frac{\sqrt{2}}{2}\right)^2 - 0^2 = \frac{1}{2}$$

23. $$\int_{-2}^{4} x^2(x^3+8)^2dx = \begin{bmatrix} x^3+8=u \\ d(x^3+8)=du \\ 3x^2dx=du \\ x^2dx=\frac{1}{3}du \end{bmatrix} = \int_{(-2)^3+8}^{4^3+8} u^2\frac{1}{3}du = \frac{1}{3}\int_0^{72} u^2du = \frac{1}{3}\left[\frac{u^3}{3}\right]_0^{72} = \frac{1}{9}72^3 = 41,472$$

24. $$\int_0^2 \frac{x}{\sqrt{2x^2+1}}dx = \begin{bmatrix} 2x^2+1=u \\ d(2x^2+1)=du \\ 4xdx=du \\ xdx=\frac{1}{4}du \end{bmatrix} = \int_{2\cdot 0^2+1}^{2\cdot 2^2+1} u^{-1/2}\frac{1}{4}du = \frac{1}{4}\int_1^9 u^{-1/2}du = \frac{1}{4}\left[\frac{u^{1/2}}{1/2}\right]_1^9 = \frac{3}{2}-\frac{1}{2} = 1$$

25. $$\int_1^9 \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}dx = \begin{bmatrix} \sqrt{x}+1=u \\ d(\sqrt{x}+1)=du \\ \frac{1}{2\sqrt{x}}dx=du \\ \frac{1}{\sqrt{x}}dx=2du \end{bmatrix} = \int_{\sqrt{1}+1}^{\sqrt{9}+1} u^{-2}2du = 2\int_2^4 u^{-2}du = \left[-\frac{2}{u}\right]_2^4 = -\frac{1}{2}+1 = \frac{1}{2}$$

26. $$\int_1^2 (x-1)\sqrt{2-x}dx = \begin{bmatrix} 2-x=u \\ 2-u=x \\ d(2-x)=du \\ -dx=du \\ dx=(-1)du \end{bmatrix} = \int_{2-1}^{2-2} -[(2-u)-1]\sqrt{u}du = \int_{2-1}^{2-2} -[1-u]\sqrt{u}du$$

$$= \int_{2-1}^{2-2} [u-1]\sqrt{u}du = \int_1^0 (u^{3/2}-u^{1/2})du$$

$$= \left[\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right]_1^0 = -\left[\frac{2}{5}-\frac{2}{3}\right] = \frac{4}{15}$$

27. $$\int_0^4 \frac{x}{\sqrt{2x+1}}dx = \begin{bmatrix} 2x+1=u \Rightarrow x=\frac{1}{2}(u-1) \\ d(2x+1)=du \\ 2dx=du \\ dx=\frac{1}{2}du \end{bmatrix} = \int_{2\cdot 0+1}^{2\cdot 4+1} \frac{1}{2}(u-1)u^{-1/2}\frac{1}{2}du = \frac{1}{4}\int_1^9 (u^{1/2}-u^{-1/2})du$$

$$= \frac{1}{4}\left[\frac{u^{3/2}}{3/2}-\frac{u^{1/2}}{1/2}\right]_1^9 = \frac{10}{3} \approx 3.333$$

28. $$\int_0^3 \left(\theta+\cos\frac{\theta}{6}\right)d\theta = \begin{bmatrix} \frac{\theta}{6}=u \Rightarrow \theta=6u \\ d\left(\frac{\theta}{6}\right)=du \\ d\theta=6du \end{bmatrix} = \int_{0/6}^{3/6} (6u+\cos u)\,6du = 36\int_0^{1/2} udu + 6\int_0^{1/2}\cos udu$$

$$= 36\left[\frac{u^2}{2}\right]_0^{1/2} + 6\left[\sin u\right]_0^{1/2} \approx 7.377$$

# Appendix I

To find $\int \frac{x+1}{x^2+x+1}dx$, we first rewrite the denominator as

$$x^2+x+1 = x^2+2x\cdot\frac{1}{2}+1 = x^2+2x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1 = \left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1$$

$$= \left(x+\frac{1}{2}\right)^2-\frac{1}{4}+1 = \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$$

We have

$$\int \frac{x+1}{x^2+x+1}dx = \int \frac{x+1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx = \left[\begin{array}{l} x+\frac{1}{2}=u \\ d\left(x+\frac{1}{2}\right)=du \\ dx=du \end{array}\right] = \int \frac{u-\frac{1}{2}+1}{u^2+\frac{3}{4}}du$$

$$= \int \frac{u+\frac{1}{2}}{u^2+\frac{3}{4}}du = \int \frac{u}{u^2+\frac{3}{4}}du + \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}du = \int \frac{u}{u^2+\frac{3}{4}}du + \frac{1}{2}\int \frac{1}{u^2+\left(\frac{\sqrt{3}}{2}\right)^2}du$$

Note that

$$\int \frac{u}{u^2+\frac{3}{4}}du = \left[\begin{array}{l} u^2+\frac{3}{4}=v \\ d\left(u^2+\frac{3}{4}\right)=dv \\ 2udu=dv \\ udu=\frac{1}{2}dv \end{array}\right] = \frac{1}{2}\int \frac{1}{v}dv = \frac{1}{2}\ln|v|+C = \frac{1}{2}\ln\left(u^2+\frac{3}{4}\right)+C$$

and

$$\int \frac{1}{u^2+a^2}du = \int \frac{1}{a^2\left(\frac{u^2}{a^2}+1\right)}du = \frac{1}{a^2}\int \frac{1}{\left(\frac{u}{a}\right)^2+1}du = \left[\begin{array}{l} \frac{u}{a}=v \\ d\left(\frac{u}{a}\right)=dv \\ \frac{1}{a}du=dv \\ du=adv \end{array}\right] = \frac{1}{a^2}\int \frac{1}{v^2+1}adv$$

$$= \frac{1}{a}\int \frac{1}{v^2+1}dv = \frac{1}{a}\arctan v + C = \frac{1}{a}\arctan\frac{u}{a}+C$$

hence

$$\int \frac{1}{u^2+\left(\frac{\sqrt{3}}{2}\right)^2}du = \frac{2}{\sqrt{3}}\arctan\left(\frac{2u}{\sqrt{3}}\right)+C$$

Therefore

$$\int \frac{x+1}{x^2+x+1}dx = \int \frac{u}{u^2+\frac{3}{4}}du + \frac{1}{2}\int \frac{1}{u^2+\left(\frac{\sqrt{3}}{2}\right)^2}du = \frac{1}{2}\ln\left(u^2+\frac{3}{4}\right)+\frac{1}{2}\cdot\frac{2}{\sqrt{3}}\arctan\left(\frac{2u}{\sqrt{3}}\right)+C$$

$$= \frac{1}{2}\ln(x^2+x+1)+\frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+C$$

# Appendix II

To find $\int \frac{dx}{1-x^2}$, we first note that

$$\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right)$$

Indeed,

$$\frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right) = \frac{1}{2}\left(\frac{1+x}{(1-x)(1+x)} + \frac{1-x}{(1-x)(1+x)}\right) = \frac{1}{2}\cdot\frac{1+x+1-x}{(1-x)(1+x)}$$

$$= \frac{1}{2}\cdot\frac{2}{(1-x)(1+x)}$$

$$= \frac{1}{(1-x)(1+x)}$$

Therefore

$$\int \frac{dx}{1-x^2} = \frac{1}{2}\int\left(\frac{1}{1-x} + \frac{1}{1+x}\right)dx = \frac{1}{2}\int\frac{1}{1-x}dx + \frac{1}{2}\int\frac{1}{1+x}dx$$

We have

$$\int \frac{1}{1-x}dx = \begin{bmatrix} 1-x=u \\ d(1-x)=du \\ -dx=du \\ dx=-du \end{bmatrix} = -\int\frac{1}{u}du = -\ln|u| + C = -\ln|1-x| + C$$

and

$$\int \frac{1}{1+x}dx = \begin{bmatrix} 1+x=u \\ d(1+x)=du \\ dx=du \end{bmatrix} = \int\frac{1}{u}du = \ln|u| + C = \ln|1+x| + C$$

It follows that

$$\int \frac{dx}{1-x^2} = = \frac{1}{2}\int\frac{1}{1-x}dx + \frac{1}{2}\int\frac{1}{1+x}dx$$

$$= -\frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x| + C$$

$$= \frac{1}{2}\left(-\ln|1-x| + \ln|1+x|\right) + C$$

$$= \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$$

# Appendix III

EXAMPLE: Find

(a) $\displaystyle\int_0^1 \frac{\sqrt{x}+1}{\sqrt{x}}dx$ (b) $\displaystyle\int_0^1 \frac{\sqrt{x}}{1+\sqrt{x}}dx$

Solution:

(a) We have

$$\int_0^1 \frac{\sqrt{x}+1}{\sqrt{x}}dx = \int_0^1 \left(\frac{\sqrt{x}}{\sqrt{x}}+\frac{1}{\sqrt{x}}\right)dx = \int_0^1 \left(1+x^{-1/2}\right)dx = \left[x+\frac{x^{-1/2+1}}{-1/2+1}\right]_0^1 = \left[x+\frac{x^{1/2}}{1/2}\right]_0^1$$

$$= \left[x+2\sqrt{x}\right]_0^1 = \left(1+2\sqrt{1}\right)-\left(0+2\sqrt{0}\right) = 3$$

(b) We have

$$\int_0^1 \frac{\sqrt{x}}{1+\sqrt{x}}dx = \left[\begin{array}{r} 1+\sqrt{x}=u \implies \sqrt{x}=u-1 \implies x=(u-1)^2 \\ dx=d(u-1)^2 \\ dx=2(u-1)du \end{array}\right] = \int_{1+\sqrt{0}}^{1+\sqrt{1}} \frac{u-1}{u}\cdot 2(u-1)du$$

$$= 2\int_1^2 \frac{(u-1)^2}{u}du = 2\int_1^2 \frac{u^2-2u+1}{u}du = 2\int_1^2 \left(\frac{u^2}{u}-\frac{2u}{u}+\frac{1}{u}\right)du = 2\int_1^2 \left(u-2+\frac{1}{u}\right)du$$

$$= 2\left[\frac{u^2}{2}-2u+\ln|u|\right]_1^2 = 2\left[\left(\frac{2^2}{2}-2\cdot 2+\ln 2\right)-\left(\frac{1^2}{2}-2\cdot 1+\ln 1\right)\right]$$

$$= 2\left[(2-4+\ln 2)-\left(\frac{1}{2}-2+0\right)\right] = 2\left[\ln 2-\frac{1}{2}\right] = 2\ln 2-1$$

We can apply the $u$-substitution in a bit different way:

$$\int_0^1 \frac{\sqrt{x}}{1+\sqrt{x}}dx = \left[\begin{array}{l} 1+\sqrt{x}=u \implies \sqrt{x}=u-1 \\ d(1+\sqrt{x})=du \\ \dfrac{1}{2\sqrt{x}}dx=du \\ dx=2\sqrt{x}du \\ dx=2(u-1)du \end{array}\right] = \int_{1+\sqrt{0}}^{1+\sqrt{1}} \frac{u-1}{u}\cdot 2(u-1)du = [\text{by the above}] = 2\ln 2-1$$

REMARK: Problem (b) was given in Fall 2013 (Calculus II, quiz 1). Nobody solved this problem correctly.